# Reliable Broadcast in a Wireless Grid Network with Probabilistic Failures

*Technical Report (October 12, 2005)*

Vartika Bhandari
Dept. of Computer Science, and
Coordinated Science Laboratory
University of Illinois at Urbana-Champaign
vbhandar@uiuc.edu

Nitin H. Vaidya
Dept. of Electrical and Computer Eng., and
Coordinated Science Laboratory
University of Illinois at Urbana-Champaign
nhv@uiuc.edu

**Abstract**

We consider a wireless grid network in which nodes are prone to failure. In the considered failure mode, each node has an independent probability of failure $p$, and failures may be either Byzantine or crash-stop in nature. All nodes are assumed to have a common transmission range $r$, and a resultant common degree $d$. We establish necessary and sufficient conditions for the degree of each node as a function of the total network size $n$ and the failure probability $p$, so as to ensure that reliable broadcast succeeds with probability 1, as $n \to \infty$. Our results indicate that reliable broadcast is asymptotically achievable with Byzantine failures if $p < \frac{1}{2}$, and the degree of each node is $\Theta\left(\frac{\ln n}{\ln \frac{1}{2p} + \ln \frac{1}{2(1-p)}}\right)$. These results exhibit similarity of form to results obtained for crash-stop failures that indicate a required degree of $\Theta\left(\frac{\ln n}{\ln \frac{1}{p}}\right)$ for $p < 1$.

## I. Introduction

We consider the problem of reliable broadcast in a wireless grid network prone to probabilistic failures. The node failures are assumed i.i.d. with probability $p$. Two separate failure types are considered, viz., Byzantine and crash-stop. We show that when nodes exhibit Byzantine failures, reliable broadcast requires that $p < \frac{1}{2}$, and the node degree must be $\Theta\left(\frac{\ln n}{\ln \frac{1}{2p} + \ln \frac{1}{2(1-p)}}\right)$ for asymptotic achievability of reliable broadcast. This may alternatively be stated as $\Theta\left(\frac{\ln n}{D(Q_{\frac{1}{2}}||P)}\right)$ where $Q_{\frac{1}{2}}$ denotes a distribution with failure probability $\frac{1}{2}$, $P$ denotes the actual distribution with failure probability $p$, and $D(Q||P)$ denotes the *relative entropy* (or Kullback-Leibler distance) between distributions $Q$ and $P$. For crash-stop failures, the problem of reliable broadcast is equivalent to connectivity. For this case, we show that node degree must be $\Theta\left(\frac{\ln n}{\ln \frac{1}{p}}\right)$ for $p < 1$, or alternatively stated, $\Theta\left(\frac{\ln n}{D(Q_1||P)}\right)$, where $Q_1$ is the distribution with failure probability 1. This report comprises two independent parts. We consider the case of Byzantine failures in the first part. In the second part, we address the issue of crash-stop failures as a connectivity question. Along with connectivity, we also obtain conditions for coverage that point toward the same expression, except for the constants involved.

This report supercedes an earlier report ”Connectivity and Coverage in Failure-Prone Wireless Grid Networks”, dated September 2005.
This research is supported in part by the National Science Foundation, and a Vodafone Graduate Fellowship.

## II. Some Useful Mathematical Results

We state some mathematical results that have been used in our proofs:

**FACT 1.** $\forall x \in [0,1] : \ln \frac{1}{1-x} \geq x$

**FACT 2.** *If* $f(n) \leq n^{\frac{1}{2}-\varepsilon}(0<\varepsilon<\frac{1}{2})$*:*

$$\lim_{n\to\infty}\left(1+\frac{f(n)}{n}\right)^n = e^{(\lim_{n\to\infty} f(n))}$$

*Proof:* Let $f(n) \leq n^{\frac{1}{2}-\varepsilon}$, where $0<\varepsilon<\frac{1}{2}$. Let $g(n) = (1+\frac{f(n)}{n})^n$. Then:

$$\begin{aligned}
\ln g = n\ln(1+\frac{f(n)}{n}) &= n\left(\frac{f(n)}{n} - \frac{1}{2}(\frac{f(n)}{n})^2 + \frac{1}{3}(\frac{f(n)}{n})^3 - \ldots.\right)[1] \\
&= n\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k}(\frac{f(n)}{n})^k = f + \sum_{k=2}^{\infty}(-1)^{k-1}\frac{1}{k}(\frac{f(n)^k}{n^{k-1}}) \\
&\leq f(n) + f(n)\sum_{k=2}^{\infty}\frac{1}{k}(\frac{f(n)}{n})^{k-1} < f(n) + f(n)\sum_{k=2}^{\infty}(\frac{1}{\sqrt{n}})^{k-1} \\
&= f(n)\left(1+\sum_{k=1}^{\infty}(\frac{1}{\sqrt{n}})^k\right) = f(n)\left(1+\frac{1}{1-\frac{1}{\sqrt{n}}}\right) \\
&\leq 2f \text{ for } n \geq 4 \\
\therefore \left(1+\frac{f(n)}{n}\right)^n &\leq e^{2f(n)} \text{ for } n \geq 4
\end{aligned}$$

$$\begin{aligned}
\ln g = n\ln(1+\frac{f(n)}{n}) = n\left(\frac{f(n)}{n} - \frac{1}{2}(\frac{f(n)}{n})^2 + \frac{1}{3}(\frac{f(n)}{n})^3 - \ldots.\right)[1] &= n\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k}(\frac{f(n)}{n})^k \\
&= f(n) + \sum_{k=2}^{\infty}(-1)^{k-1}\frac{1}{k}(\frac{f(n)^k}{n^{k-1}}) \\
\lim_{n\to\infty}\ln g = \lim_{n\to\infty} f(n) + \sum_{k=2}^{\infty}(-1)^{k-1}\frac{1}{k}(\frac{f(n)^k}{n^{k-1}}) &= \lim_{n\to\infty} f(n)
\end{aligned}$$

$$\therefore \lim_{n\to\infty} g(n) = e^{(\lim_{n\to\infty} f(n))}$$

**FACT 3.** *If $c>0$ is a positive constant independent of $n$, and $b \geq 1$ is another positive constant independent of $n$, then $\exists n_o \in \mathcal{N}$ such that:*
$1-\frac{1}{(\ln n)^b} \leq \frac{1}{n^{\frac{c}{n}}}$ *for* $n > n_o$

*Proof:*

$$\because \frac{1}{1-\frac{1}{(\ln n)^b}} \geq e^{\frac{1}{(\ln n)^b}} \text{ (from Fact 1 )}$$

$$\therefore 1-\frac{1}{(\ln n)^b} \leq e^{-\frac{1}{(\ln n)^b}} = \frac{1}{e^{\frac{1}{(\ln n)^b}}} = \frac{1}{e^{\frac{\ln n}{(\ln n)^{(b+1)}}}}$$

$$= \frac{1}{n^{\frac{1}{(\ln n)^{(b+1)}}}} \leq \frac{1}{n^{\frac{c}{n}}} \text{ for large } n$$

$$\because \exists n_o \in \mathcal{N} \text{ s.t. } \frac{1}{(\ln n)^{(b+1)}} \geq \frac{c}{n}, \forall n > n_o$$

**LEMMA 1.** *(Jogdeo & Samuels [2]) Given $X = Y_1 + Y_2 + ..., +Y_n$ where $\forall i, Y_i = Bernoulli(p_i)$, and $\sum p_i = np$, the median $m$ of the distribution is either $\lfloor np \rfloor or \lceil np \rceil$, i.e., $Pr[X \leq m] \geq \frac{1}{2}$ and $Pr[X \geq m] \geq \frac{1}{2}$.*

**Corollary 1.** *Given $X = Y_1 + Y_2 + ..., +Y_n$ where $\forall i, Y_i = Bernoulli(p)$, the median $m$ of the distribution is either $\lfloor np \rfloor or \lceil np \rceil$, i.e., $Pr[X \leq m] \geq \frac{1}{2}$ and $Pr[X \geq m] \geq \frac{1}{2}$.*

*Proof:* The proof proceeds by setting $p_1 = p_2 = ... = p_n = p$ and applying the above-stated Lemma.

**Corollary 2.** *Given $X = Y_1 + Y_2 + ..., +Y_n$ where $n$ is even, and $\forall i, Y_i = Bernoulli(p)$ where $p \geq \frac{1}{2}$, the median $m$ of the distribution satisfies $m \geq \frac{n}{2}$.*

*Proof:* We know that $m$ is either $\lfloor np \rfloor or \lceil np \rceil$. When $p = \frac{1}{2}$, $m = \frac{n}{2}$ (as $n$ is even). For $p > \frac{1}{2}$, $m \geq \lfloor np \rfloor \geq \lfloor \frac{n}{2} \rfloor = \frac{n}{2}$.

**LEMMA 2.** *(Chernoff Bound) If $X = \sum_{i=1}^{n} X_i$, where each $X_i$ is $Bernoulli(p)$, then for $0 \leq \beta \leq 1$:*

$$Pr[X \leq (1-\beta)E[X]] \leq \exp(-\frac{\beta^2}{2}E[X]) \tag{1}$$

**LEMMA 3.** *(Relative Entropy Form of Chernoff-Hoeffding Bound[3]) If $X = \sum_{i=1}^{n} X_i$, where each $X_i$ is $Bernoulli(p)$, then for $0 \leq \beta \leq 1$:*

$$Pr[X \geq \beta n] \leq e^{-n(\beta \ln \frac{\beta}{p} + (1-\beta)\ln \frac{1-\beta}{1-p})} \tag{2}$$

**LEMMA 4.** *(Sanov's Theorem[4]) If $X_1$, $X_2$,..., $X_n$ are drawn i.i.d. from alphabet $\chi$ according to $Q(x)$, then probability of $\mathbf{x}$ is given by:*

$$Q^{(n)}(\mathbf{x}) = e^{-n(H(P_x)+D(P_{\mathbf{x}}||Q))} \tag{3}$$

*where $H$ and $P$ denote the entropy and relative entropy functions (here considered w.r.t base e).*

*Also, for any distributions $P$ and $Q$, the size of type class $T(P)$ satisfies:*

$$\frac{1}{(n+1)^{|\chi|}} e^{nH(P)} \leq |T(p)| \leq e^{nH(P)} \tag{4}$$

*and, the probability of the type class $T(P)$ under $Q$ is governed by:*

$$\frac{1}{(n+1)^{|\chi|}} e^{-n(D(P||Q))} \leq Q^{(n)}(T(p)) \leq e^{-n(D(P||Q))} \tag{5}$$

**LEMMA 5.** *Suppose $S_1$ and $S_2$ are sets of Bernoulli random variables, such that $S_1 = \{I_1, I_1, ..., I_m\}$ and $S_2 = \{I_{k+1}, ..., I_{k+m}\}$, where $\forall i, I_i = Bernoulli(p)$. If $N_1 = \sum_{I_j \in S_1} I_j$ and $N_2 = \sum_{I_j \in S_2} I_j$ then:*

$$Pr[N_2 < a | N_1 < a] \geq Pr[N_2 < a] \tag{6}$$

*Proof:* We know that $S_1 \cap S_2 = \{I_{k+1}, ..., I_m\}$. Let $M_1 = \sum_{I_j \in S_1 \cap S_2} I_j$, and let $T = \sum_{I_j \in (S_2 - S_1)} I_j$. Then $M_1 = N_1 - b$ where $b = \sum_{I_j \in (S_1 - S_2)} I_j \geq 0$. Thus $N_1 < a \Rightarrow M_1 < a - b < a$. Note that $Pr[M_1 < k | M_1 < a] = \frac{Pr[M_1 < k and M_1 < a]}{Pr[M_1 < a]} \geq Pr[M_1 < k]$.

$$Pr[N_2 < a | N_1 < a] \geq P[N_2 < a | M_1 < a] = \sum_{k=0}^{a-1} Pr[M_1 < k | M_1 < a] \cdot Pr[T = a - 1 - k] \tag{7}$$

$$\geq \sum_{k=0}^{a-1} Pr[M_1 < k] \cdot Pr[T = a - 1 - k] = Pr[N_2 < a] \tag{8}$$

## III. Asymptotic Notation

We use the following asymptotic notation:

- $O(g(n)) = \{f(n) | \exists c, N_o,$ such that $f(n) \leq cg(n)$ for $n > N_o\}$
- $o(g(n)) = \{f(n) | \lim_{n \to \infty} \frac{f(n)}{g(n)} = 0\}$
- $\omega(g(n)) = \{f(n) | g(n) = o(f(n))\}$
- $\Omega(g(n)) = \{f(n) | g(n) = O(f(n))\}$
- $\Theta(g(n)) = \{f(n) | \exists c_1, c_2, N_o,$ such that $c_1 g(n) \leq f(n) \leq c_2 g(n)$ for $n > N_o\}$

Thus, wherever we have an expression involving one of the above, it implies that we can replace the asymptotic notation term with any function of that class, and the derived result would hold.

# Byzantine Failures

## IV. Network Model

We consider a network model wherein nodes are located on a two-dimensional rectangular toroidal grid (each grid unit is a $1\times 1$ square). The case of a non-toroidal grid will be briefly discussed, and does not affect our results. We designate an origin, and all nodes can be uniquely identified by their grid location $(x,y)$ w.r.t. this origin. All nodes have a common transmission radius $r$. A message transmitted by a node $(x,y)$ is heard by all nodes within distance $r$ from it (where distance is defined in terms of the particular metric under consideration, and $r$ is assumed to be an integer). The set of these nodes is termed the neighborhood of $(x,y)$.

In this paper, we consider two distance metrics: $L_\infty$ and $L_2$. The $L_\infty$ metric is the metric induced by the $L_\infty$ norm [5], such that the distance between points $(x_1,y_1)$ and $(x_2,y_2)$ is given by $\max\{|x_1-x_2|,|y_1-y_2|\}$ in the this metric. Thus $nbd(a,b)$ comprises a square of side $2r$ with its centroid at $(a,b)$, and the degree of a node is $4r^2+4r$. The $L_2$ metric is induced by the $L_2$ norm [5], and is the Euclidean distance metric. The $L_2$ distance between points $(x_1,y_1)$ and $(x_2,y_2)$ is given by $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$, and $nbd(a,b)$ comprises nodes within a circle of radius $r$ centered at $(a,b)$. The $L_\infty$ metric enables more tractable analysis, from which necessary and sufficient conditions for the $L_2$ (Euclidean) metric proceed. In Section IX, we further elaborate on how the results for the two metrics are related.

A random failure mode is assumed, wherein each node can fail with probability $p$ independently of other nodes. Failures are Byzantine in nature. However, failed nodes cannot spoof addresses or cause deliberate collisions, i.e., the MAC layer is assumed fault-free. There is an assumption that the channel is perfectly reliable, and a local broadcast is correctly received by all neighbors. Note that this idealized shared radio channel intrinsically preserves ordering of messages sent by a node, i.e., if a node transmits messages $m_1$ and $m_2$ respectively in order, they will be received in that same order by all neighbors. We call this idealized behavior the *reliable local broadcast* assumption. The same assumption underlies the results in [6] and [7] for an adversarial fault model.

## V. Related Work

Reliable broadcast in radio networks has been studied in [8], [6], [7] and [9]. Crash-stop failures are considered in [8] for finite networks comprising nodes located in a regular grid pattern and algorithms are described for efficient broadcast to the part of the network that is reachable from the source. However this work does not attempt to quantify the number of faults that render some nodes unreachable. In [6], a locally bounded model is considered, where an adversary is free to place faults, as long as no neighborhood has more thasn $t$ faults. It was shown that for a network of nodes located on an infinite grid of unit squares and having transmission radius $r$, reliable broadcast is not achievable for $t \geq \lceil \frac{1}{2}r(2r+1)\rceil$ (in both $L_\infty$ and $L_2$ metrics). This was established as an *exact threshold* in $L_\infty$ by [7], and a protocol was described that achieved the threshold. An approximate threshold was also established for the $L_2$ metric (that is tight asymptotically, and corresponds to the same fraction of a neighborhood as in $L_\infty$). A sufficient condition for reliable broadcast in general graphs with a locally bounded adversarial model was described in [10], and a simpler protocol for the grid network case was also presented. In [11], further study of the locally bounded fault model has been undertaken on arbitrary graphs. Upper and lower bounds for achievability of reliable broadcast are presented based on graph-theoretic parameters, for arbitrary graphs. However, no exact thresholds are established. It is also shown that there exist certain graphs in which algorithms that work with knowledge of topology succeed in achieving reliable broadcast, while those that lack this knowledge fail to do so.

In closely related work, [9] considers the case of message-passing and radio networks with random transient failures. In our knowledge, the results in this paper are the first for radio networks exhibiting random but permanent Byzantine failures.

## VI. Notation and Terminology

We briefly describe here notation and terminology that shall be used in this paper. Nodes can be identified by their grid location i.e. $(x,y)$ denotes the node at $(x,y)$. The neighborhood of $(x,y)$ comprises all nodes within distance $r$ of $(x,y)$ and is denoted as $nbd(x,y)$. The degree of each node is referred to as $d$. In $L_\infty$ metric, $d = 4r^2 + 4r$, while the size of a neighborhood (including the neighborhood center) is $d+1 = 4r^2+4r+1$. Thus, the minimum degree is $d_{min} = 8$, corresponding to $r = 1$. The diameter of the network (in terms of distance, and not number of hops) is referred to as $D$. If $n$ is a perfect square, $D = \sqrt{n}$. The source of the broadcast may be deemed to be situated at $(0,0)$, without affecting generality of the results. In general, we allow any node of the network to be the source (with a corresponding shift of reference coordinates). For succint description, we define a term $pnbd(x,y)$ where $pnbd(x,y) = nbd(x-1,y) \cup nbd(x+1,y) \cup nbd(x,y-1) \cup nbd(x,y+1)$. Intuitively $pnbd(x,y)$ denotes the *perturbed neighborhood* of $(x,y)$, obtained by perturbing the center of the neighborhood to one of the nodes immediately adjacent to $(x,y)$ on the grid. Besides, we use $Be(p)$ to denote a Bernoulli random variable with parameter $p$.

## VII. Necessary Conditions for Reliable Broadcast

**THEOREM 1.** *If a node has at least half faulty neighbors, it will commit to an erroneous value with probability at least* $\frac{1}{2}$.

*Proof:* If a node has at least half faulty neighbors, it cannot hope to obtain the correct value by applying a function to all their messages. The issue is whether it is capable of selecting a subset of neighbors from which it can get the correct answer with high probability. We show that this is not possible. Consider a node $u$. Denote by $\mathcal{P}(nbd(u))$ the power set of $nbd(u)$, i.e., the set of all possible subsets of neighbors. Suppose, it is known to $u$ that half or more of its neighbors are faulty. Since failures are i.i.d., we obtain that:

$$Pr[v \in nbd(u) \text{ is faulty} | nbd(u) \text{ has half+ faults}] > \frac{1}{2} \tag{9}$$

Consider any set $S \in \mathcal{P}(nbd(u))$. Then $Pr[\text{ at least half nodes in S faulty}] \geq \frac{1}{2}$ (from Lemma 1). Hence for any subset $S$, the probability of obtaining an erroneous value from $S$ is at least $\frac{1}{2}$. Iteratively sampling over many such subsets is also not useful, as on sampling a sequence of $m$ sets $S_1, S_2, ..., S_m$, the probability that at least half the $S_i$'s had half or more faults, is at least $\frac{1}{2}$.

An alternative way to view this is that corresponding to each fault configuration $C_1$ with $t \geq \frac{d}{2}$ in $nbd(u)$, there is another configuration $C_2$ with $t$ faults, such that all non-faulty nodes in $C_1$ are faulty in $C_2$, while the non-faulty nodes in $C_2$ were all faulty in $C_1$. Then, the faulty nodes can modulate their behavior so that $u$ is unable to distinguish between the case where the correct broadcast value was $v_1$ and configuration was $C_1$ and the case when the correct value was $v_2$ and the configuration was $C_2$.

**THEOREM 2.** *When failure probability* $p \geq \frac{1}{2}$, *and* $\frac{n}{d} \to \infty$ *(this happens when* $d = o(n)$*),* $\lim_{n\to\infty} Pr[\text{ reliable broadcast fails}] > \eta > 0$ *(for some positive constant* $\eta \leq 1$*). In particular, if* $\frac{n(1-p)}{d} \to \infty$*, or if* $p \geq 1 - o(\frac{1}{n})$*, then:* $\lim_{n\to\infty} Pr[\text{ reliable broadcast fails}] = 1$.

*Proof:* Suppose failure probability $p \geq \frac{1}{2}$.

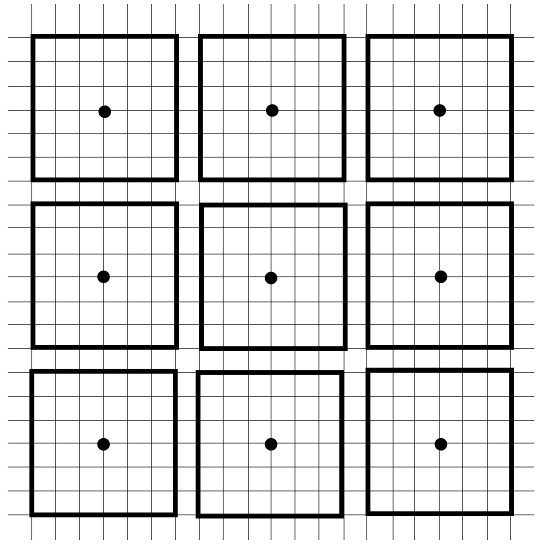

Fig. 1. Division of network into disjoint neighborhoods

*a)* $\frac{1}{2} \leq p \leq 1-\gamma (0 < \gamma < \frac{1}{2})$*:* Note that in this case, $\gamma$ can be an arbitrarily small constant, but must be independent of $n$. Consider a particular node $j$ in the network. Then, if $j$ is non-faulty, but more than half of its neighbors are faulty, reliable broadcast fails, as this node cannot get a correct view. Given that there are $d$ neighbors, and each may fail independently with probability $p$, let $Y_j$ denote the number of failed neighbors of $j$. Then, $Y$ takes values from $0,1,\ldots,d$, and $E[Y] \geq \frac{d}{2}$. Thus $\lfloor E[Y] \rfloor \geq \lfloor \frac{d}{2} \rfloor = \frac{d}{2}$ (since $d = 4r^2 + 4r$ is always even). Thus, $Pr[Y \geq \frac{d}{2}] \geq Pr[Y \geq \lfloor E[Y] \rfloor] \geq \frac{1}{2}$ (from lemma 1). Let us call this probability $q$. When $p \leq 1-\gamma$ (for arbitrarily small constant $\gamma$), we have $1-p \geq \gamma > 0$. Thus:

$$Pr[\ j \text{ alive; at least half } nbd(j) \text{ faulty }] \geq (1-p)q \geq \frac{\gamma}{2} > 0 \tag{10}$$

Let us mark out a subset of nodes $j$ such that the neighborhoods of these nodes are all disjoint, as in Fig. 1. Then the number of such nodes that we may obtain is approximately $k = \lfloor \frac{n}{d} \rfloor \geq \frac{n}{d} - 1$ (a more precise number would be $(\lfloor \frac{\sqrt{n}}{r} \rfloor)^2$ (where $d = 4r^2 + 4r$), but the loss of precision is negligible for large $n$). Let $I_j$ be an indicator variable that takes value 1 if $j$ is non-faulty but has at least half faulty neighbors. Then $Pr[I_j = 1] \geq \frac{\gamma}{2} > 0$, and all $I_j$'s are independent. Consider the case where $\frac{n}{d} \to \infty$. Let $X$ be a random variable indicating the number of non-faulty nodes with at least half faulty neighbors. Then $E[X] = \sum Pr[I_j = 1] \geq \frac{\gamma}{2}\left(\frac{n}{d} - 1\right) \to \infty$. Thus from the Chernoff Bound in Lemma 2:

$$Pr[X \leq \beta E[X]] \leq e^{-\frac{(1-\beta)^2 E[X]}{2}} (0 < \beta < 1)$$

$$\lim_{n\to\infty} Pr[X > \beta E[X]] > \lim_{n\to\infty} 1 - e^{-\frac{(1-\beta)^2 E[X]}{2}} = 1 (\because E[X] \to \infty) \tag{11}$$

Thus, as $n \to \infty$, the number of non-faulty nodes isolated by half or more faulty neighbors will also tend to infinity with probability 1.

*b)* $1-\gamma < p \leq 1 - \omega(\frac{d}{n})$*:* This is relevant if $d$ is an increasing function of $n$ and/or $p$. Once again, consider a particular node $j$ in the network. Then, if $j$ is non-faulty, but more than half of its neighbors are faulty, reliable broadcast fails. Given that there are $d$ neighbors, and each may fail independently with probability $p$, let $Y_j$ denote the number of failed neighbors of $j$. Then, $Y$ takes values from $0,1,...,d$, and $E[Y] = pd > (1-\gamma)d > \frac{1}{2}d$. We set $\beta = 1 - \frac{1}{2(1-\gamma)}$ and apply the Chernoff bound in Lemma 2. This yields:

$$Pr[Y_j \leq \frac{d}{2}) \leq \exp\left(-\frac{\left(1-\frac{1}{2(1-\gamma)}\right)^2}{2}(1-\gamma)d\right) \quad (12)$$

$$\leq \exp\left(-\frac{\left(\frac{1}{4(1-\gamma)}-\gamma\right)}{2}d\right) < \exp\left(-\frac{\frac{1}{4}}{2}d\right) \text{ if } 0 < \gamma < \frac{1}{16} \quad (13)$$

$$\leq \frac{1}{e} \text{ (as } d \geq 8) \quad (14)$$

$$\therefore Pr[Y_j \geq \frac{d}{2}] > 1 - \frac{1}{e} \quad (15)$$

Since $1-\gamma < p \leq 1 - \omega(\frac{d}{n})$, we have $\frac{n}{d}(1-p) \to \infty$. Also:

$$Pr[\ j \text{ fault-free; at least half } nbd(j) \text{ faulty }] \geq (1-p)(1-\frac{1}{e}) \quad (16)$$

Let us again mark out a subset of nodes $j$ such that the neighborhoods of these nodes are all disjoint, as in Fig. 1. Then the number of such nodes obtained is approximately $k = \lfloor \frac{n}{d} \rfloor \geq \frac{n}{d} - 1$. Let $I_j$ be an indicator variable that takes value 1 if $j$ is non-faulty but has at least half faulty neighbors. Then $Pr[I_j = 1] \geq (1-p)(1-\frac{1}{e})$, and all $I_j$'s are independent. Let $X = \sum I_j$ be a random variable denoting number of alive but isolated nodes. Then $E[X] \geq (1-p)(1-\frac{1}{e})\left(\frac{n}{d}-1\right) \approx \frac{n(1-p)(1-\frac{1}{e})}{d} \to \infty$ if $\frac{n}{d} \to \infty$. Thus from the Chernoff Bound in Lemma 2, for $0 < \beta < 1$ (e.g. $\beta = \frac{1}{3}$) :

$$Pr[X \leq \beta E[X]] \leq e^{-\frac{(1-\beta)^2 E[X]}{2}} \lim_{n\to\infty} Pr[X > \beta E[X]] > \lim_{n\to\infty} 1 - e^{-\frac{(1-\beta)^2 E[X]}{2}} = 1 (\because E[X] \to \infty) \quad (17)$$

Thus, as $n \to \infty$, the number of non-faulty nodes isolated by half or more faulty neighbors will also tend to infinity with probability 1.

*c) $1-\omega(\frac{d}{n}) < p \leq 1-\omega(\frac{1}{n})$:* Note that $n(1-p) \to \infty$. Thus, it is easily seen there will still be a large number of fault-free nodes in the network (and this number will also tend to infinity as $n$ increases). The cases of interest are those in which at least two non-neighboring nodes in the entire network are alive (else the broadcast issue is either trivialized or moot), and as $n(1-p) \to \infty$ in this range, there will indeed be at least two such alive nodes with probability $q \to 1$ (as may be verified by application of the Chernoff bound from Lemma 2). Then, consider each of these alive nodes, say $A$ and $B$. The probability that half or more of $A$'s neighbors are faulty can be no less that that in the previous case, i.e., $Pr[$ A has half or more faulty neighbors$] \geq 1-\frac{1}{e}$. Similarly, $Pr[$ B has half or more faulty neighbors$] \geq 1-\frac{1}{e}$. Then $Pr[$A or/and B has half or more faulty neighbors$] \geq 1-\frac{1}{e^2} > 0$. Hence reliable broadcast fails with a significant positive probability.

*d) $p = 1-\Theta(\frac{1}{n})$:*

$$Pr[\text{All nodes faulty;broadcast issue moot}] = p^n \quad (18)$$

$$\geq \left(1-\Theta(\frac{1}{n})\right)^n = (1-g(n))^n \text{ where } g(n) = \frac{\alpha}{n} \quad (19)$$

$$\lim_{n\to\infty} Pr[\text{All nodes faulty; broadcast issue moot}] \tag{20}$$

$$\geq \lim_{n\to\infty}(1-g(n)))^n = \left(1-\frac{\alpha}{n})\right)^n = e^{-\alpha} > 0 \text{ from Fact 2} \tag{21}$$

*e)* $p \geq 1 - o(\frac{1}{n})$ *:*

$$Pr[\text{All nodes faulty;broadcast issue moot}] = p^n \tag{22}$$

$$\geq \left(1-o(\frac{1}{n})\right)^n = (1-g(n))^n \text{ where } \frac{g(n)}{1/n} = ng(n) \to 0 \tag{23}$$

$$\lim_{n\to\infty} Pr[\text{All nodes faulty; broadcast issue moot}] \tag{24}$$

$$\geq \lim_{n\to\infty}(1-g(n)))^n = \lim_{n\to\infty}\left(1-\frac{ng(n)}{n})\right)^n \tag{25}$$

$$= e^{-\lim(ng(n))} = 1 \text{ from Fact 2} \tag{26}$$

**THEOREM 3.** *When* $p \leq \frac{1}{2} - \varepsilon(0 < \varepsilon < \frac{1}{2})$*, and node degree* $d \leq c\frac{\ln n}{\ln\frac{1}{2p}+\ln\frac{1}{2(1-p)}}$ *(for suitable constant c), reliable broadcast asymptotically fails with probability 1.*

*Proof:* Suppose failure probability $p \leq \frac{1}{2} - \varepsilon$, where $\varepsilon$ is an arbitrarily small constant. Choose an increasing function $f(n) = o(\sqrt{n})$, and a constant $0 < c < 1$ such that $\frac{c}{2}\ln n \leq \ln n - 3\ln\ln n - 2\ln f(n)$, for sufficiently large $n$. Take $n$ to be large enough so that $\ln\frac{1}{2p} + \ln\frac{1}{2(1-p)} \geq \frac{\ln n}{f(n)}$. Thus we obtain $d \leq cf(n)) < f(n)$. To illustrate, we take $f(n) = (\ln n)^2$, and $n$ to be large enough so that $\ln\frac{1}{2p} + \ln\frac{1}{2(1-p)} \geq \frac{1}{\ln n}$. Setting $d \leq c\frac{\ln n}{\ln\frac{1}{2p}+\ln\frac{1}{2(1-p)}}$, for this choice of $c$, and large enough $n$, we obtain $d \leq c(\ln n)^2 < (\ln n)^2$.

Consider a particular node $j$ in the network. Then, if $j$ is non-faulty, but more than half of its neighbors are faulty, reliable broadcast fails (from Theorem 1). Given that there are $d$ neighbors, and each may fail independently with probability $p$, let $I_{jk}(1 \leq k \leq d)$ denote the indicator variable corresponding to neighbor $k$ of $j$ (enumerated in some order), such that $I_{jk} = 1$ if $k$ is faulty, and 0 otherwise. Then $Y_j = \sum I_{jk}$ denotes the number of failed neighbors of $j$. $Y$ takes values from $0,1,...,d$, and $E[Y] = pd$. $Pr[Y_j \geq \frac{d}{2}] = \sum_{i=\frac{d}{2}}^{d}\binom{d}{i}p^i(1-p)^{(d-i)}$. Let us simply consider the event $Y_j = \frac{d}{2}$. Then we can apply the lower bound from Sanov's Theorem (Lemma 4). The variables $I_{jk}(1 \leq k \leq d)$ are drawn from $\chi = \{0,1\}$ as per distribution $Q = Be(p)$, and the distribution $P$ corresponding to $Y_j = \frac{d}{2}$ is $Be(\frac{1}{2})$ (we shall refer to this as

$Q_{\frac{1}{2}}$). $|\chi| = 2$, and $\frac{1}{(d+1)^{|\chi|}} = \frac{1}{(d+1)^2} > \frac{1}{\frac{3}{2}d^2} = \frac{2}{3}e^{-2\ln d}$ (since $d \geq 8$). Thus, we obtain:

$$Pr[Y_j \geq \frac{d}{2}] \geq Pr[Y_j = \frac{d}{2}] \geq \frac{1}{(d+1)^{|\chi|}} e^{-d(D(P||Q))} = \frac{1}{(d+1)^2} e^{-d(D(Q_{\frac{1}{2}}||Q))} > \frac{2}{3} e^{-d(D(Q_{\frac{1}{2}}||Q))-2\ln d} \tag{27}$$

$$> \frac{2}{3} e^{-(c\frac{\ln n}{\ln\frac{1}{2p}+\ln\frac{1}{2(1-p)}})(\frac{1}{2}\ln\frac{1}{2p}+\frac{1}{2}\ln\frac{1}{2(1-p)})-4\ln\ln n} \tag{28}$$

$$\text{(since } n \text{ is chosen large enough to ensure that } \ln\frac{1}{2p} + \ln\frac{1}{2(1-p)} \geq \frac{1}{\ln n}, \text{ and } c < 1, \tag{29}$$

$$\text{leading to } d \leq c(\ln n)^2 < (\ln n)^2 \text{ )} \tag{30}$$

$$= \frac{2}{3} e^{-\frac{c}{2}\ln n - 4\ln\ln n} \geq \frac{2(\ln n)^3}{3n} \text{ from our choice of } c \tag{31}$$

Let us call this probability $q$.

$$Pr[\ j \text{ alive; at least half } nbd(j) \text{ faulty }] \geq (1-p)q \tag{32}$$

$$> \frac{1}{2}\frac{2(\ln n)^3}{3n} = \frac{(\ln n)^3}{3n} \tag{33}$$

Let us mark out a subset of nodes $j$ such that the neighborhoods of these nodes are all disjoint, as in Fig. 1. Then the number of such nodes that we may obtain is approximately $k = \lfloor \frac{n}{d} \rfloor \geq \frac{n}{d} - 1$. Let $I_j$ be an indicator variable that takes value 1 if $j$ is non-faulty but has at least half faulty neighbors. Then $Pr[I_j = 1] \geq \frac{(\ln n)^3}{3n}$, and all $I_j$'s are independent. Consider the case where $\frac{n}{d} \to \infty$, as $n \to \infty$. We have chosen $n$ large enough to ensure that $\ln\frac{1}{2p} + \ln\frac{1}{2(1-p)} \geq \frac{1}{\ln n}$, i.e. $d \leq c(\ln n)^2$. Let $X$ be a random variable indicating the number of non-faulty nodes with half or more faulty neighbors. Then $\sum I_j$, and $E[X] = \sum Pr[I_j = 1] \geq \frac{(\ln n)^3}{3n}\left(\frac{n}{d} - 1\right) \approx \frac{(\ln n)^3}{3d} > \frac{\ln n}{3} \to \infty$ (as $d < (\ln n)^2$). Thus we can apply the Chernoff bound in Lemma 2 to obtain:

$$Pr[X \leq \beta E[X]] \leq e^{-\frac{(1-\beta)^2 E[X]}{2}} \tag{34}$$

$$\lim_{n\to\infty} Pr[X > \beta E[X]] > \lim_{n\to\infty} 1 - e^{-\frac{(1-\beta)^2 E[X]}{2}} = 1 \because E[X] \to \infty \tag{35}$$

Thus, as $n \to \infty$, the number of non-faulty nodes isolated by half or more faulty neighbors will also tend to infinity with probability 1.

$$\lim_{n\to\infty} Pr[\text{ reliable broadcast fails}] \to 1$$

## VIII. Sufficient Condition for Reliable Broadcast

We now present a sufficient condition for the asymptotic achievability of reliable broadcast.

**THEOREM 4.** *When $p < \frac{1}{2}$, and node degree $d \geq \max\{d_{min}, 16\frac{\ln n}{\ln\frac{1}{p}+\ln\frac{1}{2(1-p)}}\} = \max\{d_{min}, 16\frac{\ln n}{D(Q_{\frac{1}{2}}||P)})\}$ (recall that $d_{min} = 8$ corresponding to $r = 1$), reliable broadcast is asymptotically achievable with probability 1.*

Note that when $\ln\frac{1}{2p} + \ln\frac{1}{2(1-p)} \leq \frac{16\ln n}{n}$, the degree exceeds total network size $n$, and thus the sufficient condition ceases to be relevant, merely indicating that having a single-hop network suffices for reliable broadcast (which is the trivial sufficient condition for the assumed radio network model). Thus the sufficient condition is of interest only so long as $\ln\frac{1}{2p} + \ln\frac{1}{2(1-p)} > \frac{16\ln n}{n}$.

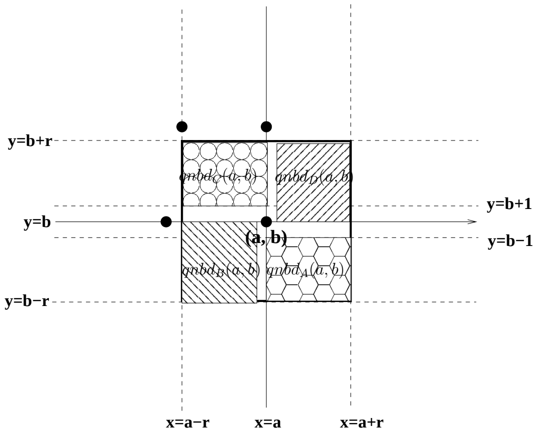


Fig. 2. Depiction of $qnbd_A$, $qnbd_B$, $qnbd_C$, $qnbd_D$

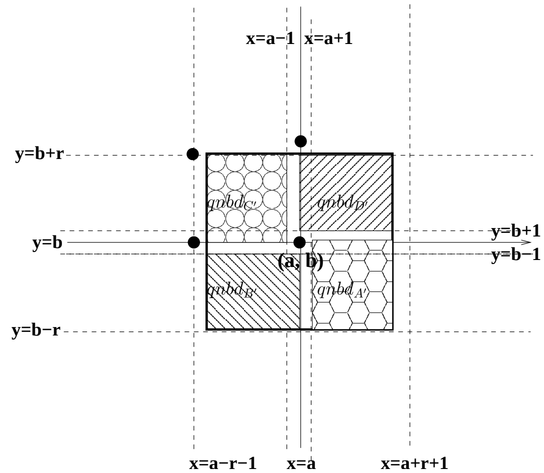


Fig. 3. Depiction of $qnbd_{A'}$, $qnbd_{B'}$, $qnbd_{C'}$, $qnbd_{D'}$

| **Region** | **x-extent** | **y-extent** |
|---|---|---|
| $qnbd_A(a,b)$ | $a \leq x \leq (a+r)$ | $(b-r) \leq y \leq (b-1)$ |
| $qnbd_B(a,b)$ | $(a-r) \leq x \leq (a-1)$ | $(b-r) \leq y \leq b$ |
| $qnbd_C(a,b)$ | $(a-r) \leq x \leq a$ | $(b+1) \leq y \leq (b+r)$ |
| $qnbd_D(a,b)$ | $(a+1) \leq x \leq (a+r)$ | $b \leq y \leq (b+r)$ |
| $qnbd_{A'}(a,b)$ | $(a+1) \leq x \leq (a+r)$ | $(b-r) \leq y \leq b$ |
| $qnbd_{B'}(a,b)$ | $(a-r) \leq x \leq a$ | $(b-r) \leq y \leq (b-1)$ |
| $qnbd_{C'}(a,b)$ | $(a-r) \leq x \leq (a-1)$ | $b \leq y \leq (b+r)$ |
| $qnbd_{D'}(a,b)$ | $a \leq x \leq (a+r)$ | $(b+1) \leq y \leq (b+r)$ |

TABLE I

SPATIAL EXTENTS OF QUARTER NEIGHBORHOODS

*a) $p \leq o(\frac{1}{n})$:* When the failure probability is so small as to fall in this range, the probability of even a single node failing approaches 0 asymptotically, and thus reliable broadcast is trivially ensured even with the minimum transmission range of 1. This may be seen thus:

$$Pr[\text{No failures;trivial broadcast}] = (1-p)^n \tag{36}$$

$$\geq \left(1 - o(\frac{1}{n})\right)^n \tag{37}$$

$$\lim_{n\to\infty} Pr[\text{No failures;trivial broadcast}] \tag{38}$$

$$\geq \lim_{n\to\infty} \left(1 - o(\frac{1}{n})\right)^n = e^{-\lim(no(\frac{1}{n}))} = 1 \text{ from Fact 2} \tag{39}$$

*b) $p = \Omega(\frac{1}{n})$:* We define a term called quarter-neighborhood of a node $(x,y)$, and denote it by $qnbd(x,y)$. We associate eight quarter-neighborhoods with each node: $qnbd_A$, $qnbd_B$, $qnbd_C$, $qnbd_D$, $qnbd_{A'}$, $qnbd_{B'}$, $qnbd_{C'}$, $qnbd_{D'}$. The quarter-neighborhoods for a node $(a,b)$ are depicted in Fig. 2 and 3, and their spatial extents are tabulated in Table I. Observe that $qnbd_B(a,b) = qnbd'_A(a-r-1,b)$, $qnbd_C(a,b) = qnbd_A(a-r,b+r+1)$, and $qnbd_D(a,b) = qnbd'_A(a,b+r+1)$. Similarly, $qnbd_{B'}(a,b) = qnbd_A(a-r-1,b)$, $qnbd_{C'}(a,b) = qnbd_{A'}(a-r-1,b+r)$, and $qnbd_{D'}(a,b) = qnbd_A(a,b+r+1)$ Thus if we simply consider $qnbd_A(u)$ and $qnbd_{A'}(u) \forall$ nodes $u$, we will have considered all quarter-neighborhoods, i.e. the number of distinct (but *not disjoint*) quarter-neighborhoods is $2n$. Henceforth, we shall sometimes use $Q(x,y)$ to refer to $qnbd_A(x,y)$, and $Q'(x,y)$ to refer to $qnbd_{A'}(x,y)$. The population of any *qnbd* is $r(r+1)$, and since $d = 4r^2+4r = 4r(r+1)$, the *qnbd* population = $\frac{d}{4}$. We now state and prove the following result which is crucial to proving our sufficient condition for reliable broadcast:

**THEOREM 5.** *If* $p < \frac{1}{2}$, $d \geq \max\{d_{min}, 16\frac{\ln n}{\ln\frac{1}{2p}+\ln\frac{1}{2(1-p)}}\} = \max\{d_{min}, 16\frac{\ln n}{D(Q_{\frac{1}{2}}||P)})\}$, *then:*

$$\lim_{n\to\infty} Pr[\ \forall(x,y) \text{ less than } \frac{d}{8} \text{ faults in } Q(x,y) \text{ and } Q'(x,y)] \to 1$$

*Proof:* As shown above, the population of any *qnbd* is $\frac{d}{4}$. Each node may fail independently with probability $p$.Let $Y_{(x,y)}$ be a random variable denoting the number of faulty nodes in $Q(x,y)$. Then $E[Y_{(x,y)}] = p\frac{d}{4}$. Using $\delta = \frac{1}{2p} - 1$, we may then apply the relative entropy form of the Chernoff bound (Lemma 3) to $Y_{(x,y)} = \sum_{j\in nbd(x,y)} I_j$. Note that $d \geq \max\{d_{min}, 16\frac{\ln n}{\ln\frac{1}{2p}+\ln\frac{1}{2(1-p)}}\} \geq 16\frac{\ln n}{\ln\frac{1}{2p}+\ln\frac{1}{2(1-p)}}$. Thus, we obtain:

$$Pr[Y_{(x,y)} \geq \frac{d}{8}] \leq e^{-\frac{d}{4}(\frac{1}{2}\ln\frac{1}{2p}+\frac{1}{2}\ln\frac{1}{2(1-p)})} \tag{40}$$

$$\leq e^{-(\frac{16\ln n}{4(\ln\frac{1}{2p}+\ln\frac{1}{2(1-p)}})))(\frac{1}{2}\ln\frac{1}{2p}+\frac{1}{2}\ln\frac{1}{2(1-p)})} \tag{41}$$

$$= e^{-2\ln n} = \frac{1}{n^2} \tag{42}$$

Similarly, setting $Y'_{(x,y)}$ be a random variable denoting the number of faulty nodes in $Q'(x,y)$, we obtain that:

$$Pr[Y'_{(x,y)} \geq \frac{d}{8}] \leq \frac{1}{n^2} \tag{43}$$

The $Y_{(x,y)}$'s and $Y'_{(x,y)}$'s are not independent, as they are not all disjoint. However, it may be seen that where dependence exists, it is that of positively correlation (Lemma 5). Thus $Pr[Y_{(x',y')} < \frac{d}{8}|Y_{(x,y)} < \frac{d}{8}] \geq Pr[Y_{(x',y')} < \frac{d}{8}]$, and $Pr[Y_{(x',y')} < \frac{d}{8}|Y'_{(x,y)} < \frac{d}{8}] \geq Pr[Y_{(x',y')} < \frac{d}{8}]$. Similarly, we obtain that: $Pr[Y'_{(x',y')} < \frac{d}{8}|Y_{(x,y)} < \frac{d}{8}] \geq Pr[Y'_{(x',y')} < \frac{d}{8}]$, and $Pr[Y'_{(x',y')} < \frac{d}{8}|Y'_{(x,y)} < \frac{d}{8}] \geq Pr[Y'_{(x',y')} < \frac{d}{8}]$ Hence:

$$Pr[\forall(x,y), Y(x,y) < \frac{d}{8} \text{ and } Y'(x,y) < \frac{d}{8}] \tag{44}$$

$$\geq \prod Pr[Y_{(x',y')} < \frac{d}{8}] \prod Pr[Y'_{(x',y')} < \frac{d}{8}] \tag{45}$$

$$= \left(1 - \frac{1}{n^2}\right)^n \left(1 - \frac{1}{n^2}\right)^n \tag{46}$$

$$= \left(1 - \frac{1}{n^2}\right)^2 n \tag{47}$$

$$\therefore \lim_{n\to\infty} Pr[\forall(x,y), Y(x,y) < \frac{d}{8} \text{ and } Y'(x,y) < \frac{d}{8}] \tag{48}$$

$$\geq \lim_{n\to\infty}\left(1 - \frac{1}{n^2}\right)^2 n = e^{-\lim(\frac{2}{n})} = 1 \text{ from Fact 2} \tag{49}$$

We now consider a simple broadcast protocol that is similar to the protocol described in [6] for the adversarial model:

- Initially, the source does a local broadcast of the message.
- Each neighbor $i$ of the source immediately commits to the the first value $v$ it heard from the source, and then locally broadcasts it once in a $COMMITTED(i,v)$ message.
- Hereafter, the following protocol is followed by each node $j$ (including those involved in the previous two steps):
  If $\frac{1}{2}r(r+1)+1 = \frac{d}{8}+1$ $COMMITTED(i,v)$ message are received for a certain value $v$, from neighbors $i$ all lying within a single *qnbd*, commit to $v$, and locally broadcast a $COMMITTED(j,v)$ message.

**THEOREM 6.** (Probabilistic Correctness) *The probability that a node shall commit to a wrong value by following the above protocol diminishes to 0 asymptotically.*

*Proof:* If all $Q(x,y)(Q'(x,y))$ have strictly less than $\frac{d}{8}$ faults, the correctness of the protocol proceeds as follows:

The proof is by contradiction. Consider the first fault-free node, say $j$, that makes a wrong decision to commit to a value $v$. This implies that $\frac{d}{8}+1$ of its neighbors within some *qnbd* broadcast a $COMMITTED$ message for $v$ (the $COMMITTED$ messages were directly heard, leaving no place for doubt). All of these nodes cannot be faulty, as no more than $\frac{d}{8}$ nodes in any *qnbd* are faulty. Thus there was at least one fault-free node that committed to $v$. Since $j$ is the first fault-free node to make a wrong decision, none of the fault-free nodes amongst the $\frac{d}{8}+1$ nodes could have made a wrong decision. Thus $v$ must indeed be the correct value.

We know that all $Qnbd(x,y)$ have less than $\frac{d}{8}$ faults with probability 1 asymptotically, and hence the protocol also functions correctly with probability 1 asymptotically.

**THEOREM 7.** (Probabilistic Completeness) *Each node is eventually able to commit to the (probabilistically) correct value.*

*Proof:*
The proof proceeds by induction.

*Base Case:*

All honest nodes in $nbd(0,0)$ are able to commit to the correct value. This follows trivially since they hear the origin directly, and we assume that address-spoofing is impossible.

*Inductive Hypothesis:*

If all honest neighbors of a node located at $(a,b)$ i.e. all honest nodes in $nbd(a,b)$ are able to commit to the correct value, then all honest nodes in $pnbd(a,b)$ are able to commit to the correct value.

*Proof of Inductive Hypothesis:*

We show that each node $P$ in $pnbd(a,b) - nbd(a,b)$ has one of $qnbd_A(P)$, $qnbd_B(P)$, $qnbd_C(P)$, $qnbd_D(P)$, $qnbd_{A'}(P)$, $qnbd_{B'}(P)$, $qnbd_{C'}(P)$, $qnbd_{D'}(P)$ fully contained in $nbd(a,b)$. Since no more than $\frac{d}{8}$ of the nodes in a *qnbd* are faulty with probability 1 (asymptotically), this guarantees that the node

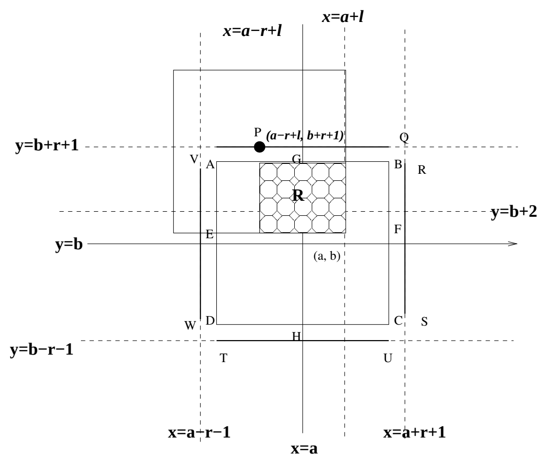


Fig. 4. Node at P has a *qnbd* in $nbd(a,b)$

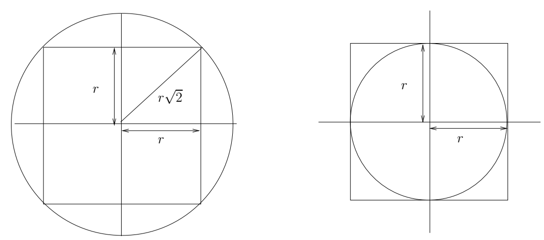


Fig. 5. Relationship between $L_\infty$ and $L_2$ neighborhoods

will become aware of $\frac{d}{8}+1$ nodes in $nbd(a,b)$ having committed to a (the correct) value, and will also commit to it. The situation is depicted in Fig. 4 for $P \in \{(a-r+l, b+r+1) | 1 \leq l \leq r\}$, for which $qnbd_A(P)$ lies in $nbd(a,b)$. For all other locations, a similar argument holds.

## IX. Conditions in Euclidean Metric

We show that our results derived for $L_\infty$ metric continue to hold for $L_2$ metric, with only the constants in the theta notation changing.

**LEMMA 6.** *If reliable broadcast is achievable asymptotically in $L_\infty$ for all $r \geq r_{min}$, then it is achievable asymptotically in $L_2$ for all $r \geq r_{min}\sqrt{2}$.*

*Proof:* The proof is by contradiction. Suppose that, for a given failure configuration, broadcast is asymptotically achievable in $L_\infty$ for all $r \geq r_{min}$ but is not asymptotically achievable for all $r \geq r_{min}\sqrt{2}$ in $L_2$. Observe that it is possible to circumscribe a $L_\infty$ neighborhood of range $r$ by a $L_2$ neighborhood of range $r\sqrt{2}$ (Fig. 5). Hence the non-faulty nodes in an $L_2$ network of transmission range $r\sqrt{2}$ can be made to simulate the operation of nodes in a $L_\infty$ network with range $r$ (as the $L_\infty$ neighborhood is fully contained within the $L_2$ neighborhood). Also, given that this is a network of known topology, with no address spoofing allowed, the faulty nodes cannot gain any unfair advantage, by not simulating the the $L_\infty$ network. This implies that if broadcast is achievable in the $L_\infty$ network of range $r$ , so must it be in the $L_2$ network of range $r\sqrt{2}$. If there is some $r \geq r_{min}$ for which we can achieve broadcast in the $L_\infty$ network asymptotically, but not in the the $L_2$ network of range $r\sqrt{2}$, we obtain a contradiction, as achievability in the $L_\infty$ network would imply achievability in the $L_2$ network.

**LEMMA 7.** *If reliable broadcast fails asymptotically in $L_\infty$ for all $r \leq r_{min}$, then it fails asymptotically in $L_2$ for all $r \leq r_{min}$.*

*Proof:* The proof is by contradiction. Suppose that broadcast fails asymptotically in $L_\infty$ for range $r$, but does not fail in $L_2$ for range $r$. Observe that an $L_\infty$ neighborhood of transmission range $r$ circumscribes an $L_2$ neighborhood of range $r$ (Fig. 5). Thus, for any given failure configuration, if broadcast succeeds in the the $L_2$ network of range $r$, so can it in the $L_\infty$ network of radius $r$, as we could simply make the fault-free nodes in the $L_\infty$ network simulate the behavior of nodes in the $L_2$ network. Hence, if broadcast does not fail in the $L_2$ network of range $r \leq r_{min}$, it will not fail in the $L_\infty$ network of range $r \leq r_{min}$. This yields a contradiction.

## X. Non-Toroidal Networks

We used the assumption that the network is toroidal to avoid edge effects. However, one can see that the results would continue to hold even if the network were spread over a non-toroidal rectilinear domain. The necessary condition would continue to hold, since the degree of nodes at the edges can be no more more than the degree of nodes towards the center, and if reliable broadcast is impossible even with the assumption of equal degree for all nodes, it must certainly be impossible when some nodes (those at the edges) have a smaller degree.

The sufficient condition continues to hold since the described protocol relies on information from quarter-neighborhoods, and it can be seen that even the nodes at the edges have at least one quarter-neighborhood within the network region.

# Crash-Stop Failures/Connectivity and Coverage

## XI. Network Model

We consider a network model wherein nodes are located on a two-dimensional rectangular toroidal grid (each grid unit is a $1 \times 1$ square). The case of a non-toroidal grid will be briefly discussed, and does not affect our results. We designate an origin, and all nodes can be uniquely identified by their grid location $(x,y)$ w.r.t. this origin. All nodes have a common transmission radius $r$. A message transmitted by a node $(x,y)$ is heard by all nodes within distance $r$ from it (where distance is defined in terms of the particular metric under consideration, and $r$ is assumed to be an integer). The set of these nodes is termed the neighborhood of $(x,y)$.

In this paper, we consider two distance metrics: $L_\infty$ and $L_2$. The $L_\infty$ metric is the metric induced by the $L_\infty$ norm [5], such that the distance between points $(x_1,y_1)$ and $(x_2,y_2)$ is given by $\max\{|x_1-x_2|,|y_1-y_2|\}$ in the this metric. Thus $nbd(a,b)$ comprises a square of side $2r$ with its centroid at $(a,b)$, and the degree of a node is $4r^2+4r$. The $L_2$ metric is induced by the $L_2$ norm [5], and is the Euclidean distance metric. The $L_2$ distance between points $(x_1,y_1)$ and $(x_2,y_2)$ is given by $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$, and $nbd(a,b)$ comprises nodes within a circle of radius $r$ centered at $(a,b)$. The $L_\infty$ metric enables more tractable analysis, from which necessary and sufficient conditions for the $L_2$ (Euclidean) metric proceed. In Section IX, we further elaborate on how the results for the two metrics are related.

A random failure mode is assumed, wherein each node can fail with probability $p$ independently of other nodes. A failed node simply stops functioning, i.e., failures are of the crash-stop kind.

## XII. Related Work

Conditions for connectivity and coverage have been formulated in the context of different network models. In [12], it was proved that in a unit area network with uniformly distributed node placement, where nodes have a common transmission radius $r$, such that $\pi r^2 = \frac{(\log n + c(n))}{n}$, the network is asymptotically connected with probability one iff $c(n) \to \infty$. In [13], an alternate model was considered whereby randomly deployed nodes may modulate their transmission power (and hence range) to ensure that they have a certain number of neighbors. It was proved that each node must be connected to $\Theta(\log n)$ neighbors for asymptotic connectivity with probability one. Recently, necessary and sufficient conditions for asymptotic connectivity in a network with low duty cycle sensors have been formulated in [14].

A grid network model was considered in [15] where nodes are located at grid locations on a square grid, but may fail independently. Nodes have a common transmission range $r$. The probability of not failing is specified as $p$, and it is shown that a sufficient condition for connectivity and coverage is that transmission range $r$ must be set to ensure that node degree is $c_1(\frac{\log n}{p})$ (for some constant $c_1$). It is also shown that a necessary condition for coverage (and hence for joint coverage and connetivity) is that node degree be at least $c_2(\frac{\log n}{p})$ (for another constant $c_2$. A fallacy in the above necessary condition was pointed out by [16], and a subsequent correction [17] by the authors of [15] presents examples illustrating that the necessary condition may fail to hold for certain subranges of $p$. The issue of coverage has been examined in detail in [16] for random, grid, and poisson deployments. However, the necessary and sufficient conditions formulated by them take a more complex form, and do not point to a single $f(n,p)$ such that a degree of $\Theta(f(n,p))$ is both necessary and sufficient for asymptotic coverage. Besides, the necessary condition is formulated for the specific case when $\lim_{n\to\infty} p \to 0$

Our results are closely related to the results of [15]. However, we prove that, given a *failure* probability $p$, it is necessary and sufficient to have a degree of $\Theta(\frac{\log n}{\log \frac{1}{p}})$ for both connectivity and coverage. Expressed in the notation of [15], we stipulate a degree of $\Theta(\frac{\log n}{\log \frac{1}{1-p}})$. Our results diverge considerably from those of [15] when the failure probability becomes extremely small, and thus our necessary conditions would hold in a certain subdomain where that of [15] would not. However, there is a small sub-domain of $p$ in which our necessary conditions also cease to hold, as with the conditions of [15]. Besides, we work in the $L_\infty$ distance metric, and then map the results to $L_2$. This yields much simpler proofs. We also remark that our joint sufficient condition for connectivity and coverage is actually sufficient for 9-coverage and not merely 1-coverage (where $k$-coverage implies that each point is covered by at least $k$ non-faulty nodes). It is noteworthy that our results may be derived from analysis presented in [18] regarding the feasible rate in a sensor network, although no statement has been made in [18] in this regard.

## XIII. Notation and Terminology

We briefly describe here notation and terminology that shall be used in this paper. Nodes can identified by their grid location i.e. $(x,y)$ denotes the node at $(x,y)$. The neighborhood of $(x,y)$ comprises all nodes within distance $r$ of $(x,y)$ and is denoted as $nbd(x,y)$. The degree of each node is referred to as $d$. In $L\_\infty$ metric, $d = 4r^2 + 4r$, while the size of a neighborhood (including the neighborhood center) is $d + 1 = 4r^2 + 4r + 1$. The diameter of the network (in terms of distance, and not number of hops) is referred to as $D$. If $n$ is a perfect square, $D = \sqrt{n}$.

## XIV. Necessary Condition for Connectivity

**THEOREM 8.** *When $p < 1 - \frac{1}{\ln n}$, then in the $L_\infty$ metric, the transmission range $r$ must satisfy $r \geq \max\{1, \Omega(\sqrt{\frac{\ln n}{\ln \frac{1}{p}}})\}$, i.e., the node degree $d \geq \max\{1, \Omega(\frac{\ln n}{\ln \frac{1}{p}})\}$, else $\lim_{n\to\infty} Pr[\text{ disconnection}] = 1$.*

*Proof:* It is obvious that the minimum transmission range required for connectivity is 1, else the degree of all nodes is 0 (except in the case when connectivity loses meaning as all nodes are faulty, and so the network can be deemed connected trivially). Similarly, the network is trivially connected if $r = D$, as all nodes are in direct range of each other. Suppose that $r = \sqrt{\frac{c \ln n}{\ln \frac{1}{p}}}$. Thus, when $p \geq \frac{1}{n^{\frac{c}{n}}}$, $r = \sqrt{\frac{c \ln n}{\ln \frac{1}{p}}} \geq \sqrt{n} \geq D$, and the necessary condition ceases to be relevant (as $r = D$ ensures connectivity). We show that the network is asymptotically disconnected with probability 1 if $r < \sqrt{\frac{c \ln n}{\ln \frac{1}{p}}}$, for some constant $0 < c < 1$, as long as $p < 1 - \frac{1}{\ln n}$. Note that if $p < 1 - \delta$ for any arbitrarily small constant $\delta > 0$ (independent of $n$), then for sufficiently large $n$, the necessary condition would hold for all $p$. Also note that $1 - \frac{1}{\ln n} < \frac{1}{n^{\frac{c}{n}}}$ for large $n$ (from Fact 3). Thus, the values of $p$ for which our necessary condition holds are those in which the transmission range remains less than $D$. When $p \geq 1 - \frac{1}{n^{1+\varepsilon}}$, all nodes are faulty with probability approaching 1, and the issue of connectivity is moot. When $p \leq \frac{1}{n^c}$, $r = \sqrt{\frac{c \ln n}{\ln \frac{1}{p}}} \leq 1$, and for this range of $p$, the necessary condition lapses to having the minimum range of 1.

*a) $p \leq 1 - \frac{1}{\ln n}$:* Consider a particular node $j$ in the network. Then, if $j$ is non-faulty, but all its neighbors are faulty, we have a potential disconnection event. Given that there are $d$ neighbors, and each may fail independently with probability $p$, the probability that $j$ does not fail, but all nodes in $nbd(j)$ fail, is $(1-p)p^d$. We choose a constant $0 < c < 1$ such that $c \ln n \leq \ln n - 4 \ln \ln n$, for sufficiently large

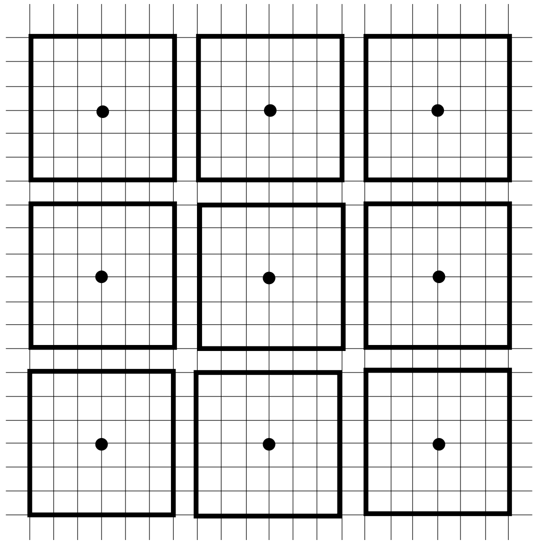

Fig. 6. Nodes having disjoint neighborhoods

$n$. In general, $c$ can be chosen very close to 1, e.g., $1-\varepsilon(0<\varepsilon<1)$, and the condition will hold for $n>n_o$, for some $n_o$. Since $p \leq 1-\frac{1}{\ln n}$, we obtain that $\frac{1}{1-p} \leq \ln n$. Let $r \leq \sqrt{\frac{c \ln n}{8 \ln \frac{1}{p}}}$. The node degree $d = 4r^2+4r \leq 4r^2+4r^2 = 8r^2$, for $n \geq 1$. Thus, for our choice of $r$, it turns out that $d \leq c\frac{\ln n}{\ln \frac{1}{p}}$. Then, it may be seen that:

$$Pr[\text{ A given node } j \text{ is alive, but isolated}] \tag{50}$$
$$\geq Pr[j \text{ is alive and all neighbors of } j \text{ are faulty }] \tag{51}$$
$$= (1-p)p^d > \frac{1}{\ln n} p^{c\frac{\ln n}{\ln \frac{1}{p}}} \tag{52}$$
$$= \frac{1}{\ln n}\frac{1}{n^c} = \frac{1}{n^c \ln n} \tag{53}$$
$$\geq \frac{(\ln n)^3}{n} \text{ (from our choice of } c) \tag{54}$$

Let us mark out a subset of nodes $j$ such that the neighborhoods of these nodes are all disjoint, as in Fig. 6. Then the number of such nodes that we may obtain = $\lfloor\left(\frac{\sqrt{n}}{2r+1}\right)^2\rfloor \geq \frac{n}{9r^2}-1$ (since $\sqrt{n}$ may not be multiple of $2r+1$). Let $I_j$ be an indicator variable that takes value 1 if $j$ is alive but isolated. Then $Pr[I_j = 1] \geq \frac{(\ln n)^3}{n}$, and all $I_j$'s are independent. Let $X$ be a random variable denoting the number of nodes from the chosen set that are alive and isolated. Then $X = \sum I_j$, and $E[X] \geq \frac{(\ln n)^3}{n}\left(\frac{n \ln \frac{1}{p}}{9c \ln n}-1\right) \geq \frac{(\ln n)^3}{n}\frac{n \ln \frac{1}{1-\frac{1}{\ln n}}}{9 \ln n} \geq \frac{1}{9}(\ln n)^2 \ln \frac{1}{1-\frac{1}{\ln n}} \geq \frac{1}{9}\ln n \to \infty$. We can thus apply the Chernoff bound from Lemma 2: Thus, with suitable $0<c<1$ and $\beta = \frac{E[X]-1}{E[X]}$, we obtain that for $p < 1-\frac{1}{\ln n}$, if $r \leq \sqrt{\frac{c \ln n}{8 \ln \frac{1}{p}}}$, then $E[X] \to \infty$, and hence $\lim_{n\to\infty} Pr[\text{ At least two alive nodes are isolated}] = 1$.

Observe that actually the necessary condition would hold for all $p$ such that $E[X] \to \infty$. For instance, the above analysis holds for all $p \leq 1-\frac{1}{(\ln n)^b}$ (where $b$ is a constant), with a corresponding suitably varying choice of $c$ to ensure that $Pr[I_j = 1] \geq \frac{(\ln n)^{(b+2)}}{n}$. Besides, if $E[X] \to \gamma > 0$, the asymptotic disconnection probability is still a positive finite quantity, and the condition is still necessary for asymptotic connectedness probability to approach 1.

*b) $p \geq 1 - \frac{1}{n^{1+\varepsilon}}$:* When the failure probability becomes so high as to fall in this range, we obtain:

$$\lim_{n\to\infty} Pr[\text{ Any node is alive}] = 1 - p^n \tag{55}$$

$$= \lim_{n\to\infty} 1 - \left(1 - \frac{1}{n^{1+\varepsilon}}\right)^n = 1 - e^{-\lim(\frac{1}{n^{\varepsilon}})} = 0 \text{ from Fact 2} \tag{56}$$

Thus the network is trivially connected by definition, regardless of degree.

## XV. Necessary Condition for Coverage

We now show that for the network to be asymptotically covered with probability approaching 1, it is necessary that the transmission range $r$ satisfy: $r \geq \max\{1, \Omega(\sqrt{\frac{\ln n}{\ln \frac{1}{p}}})\}$, i.e., the node degree be $d \geq \max\{1, \Omega(\frac{\ln n}{\ln \frac{1}{p}})\}$.

**THEOREM 9.** *For $p < 1 - \frac{1}{\ln n}$, for a suitable constant $0 < c < 1$, if $d < c\frac{\ln n}{\ln \frac{1}{p}}$ :*

$$\lim_{n\to\infty} Pr[\textit{Some point is not covered}] \to 1$$

*Proof:* As in the case of connectivity it is obvious that $r$ must be at least 1, else some points will not be covered. We handle two subranges of $p$ separately.

*a) $p < 1 - \frac{1}{\ln n}$:* The proof relies on subdivision of the network into disjoint neighborhoods, as in Fig. 6. If there exists at least one neighborhood with absolutely no nodes alive (neither the neighborhood center nor its neighbors), then the center of that neighborhood is not covered. Thus we seek to determine the probability of such an event.

We choose a constant $0 < c < 1$ such that $\frac{9}{8}c\ln n \leq \ln n - 3\ln\ln n$, for sufficiently large $n$. This ensures that $\frac{1}{n^c} \geq \frac{(\ln n)^3}{n}$ for large $n$. Let $r \leq \sqrt{\frac{c\ln n}{8\ln\frac{1}{p}}}$. The neighborhood population is given by $d + 1 = 4r^2 + 4r + 1 \leq 4r^2 + 4r^2 + r^2 = 9r^2$, for $n \geq 1$. Thus, $d + 1 \leq \frac{9}{8}c\frac{\ln n}{\ln\frac{1}{p}}$. Let $I_j$ be an indicator variable that takes value 1 if there is no alive node in the neighborhood centered at node $j$, and value 0 otherwise. Then $Pr[X_j = 1] = p^{d+1} = p^{\frac{9}{8}c\frac{\ln n}{\ln\frac{1}{p}}} = \frac{(\ln n)^3}{n}$ (from our choice of $c$). Let $X = \sum I_j$ be a random variable indicating the number of neighborhoods with no alive node. Then $E[X] = \frac{(\ln n)^3}{9r^2} = \frac{8(\ln n)^2\ln\frac{1}{p}}{9c}$ (after plugging in the chosen value of $r$). If $p < 1 - \frac{1}{\ln n}$, then $E[X] \geq \ln n(\ln n \ln\frac{1}{1-\frac{1}{\ln n}}) > \ln n \to \infty$ (from Fact 1), and application of the Chernoff bound from Lemma 2 yields that $Pr[X = 0] \leq \exp(-\frac{E[X]}{2}) \to 0$. Thus there is some uncovered region with probability 1.

Similar to the necessary condition for connectivity, observe that this necessary condition would hold for all $p$ such that $E[X] \to \infty$. In particular, the above analysis holds for all $p \leq 1 - \frac{1}{(\ln n)^b}$ (where $b$ is a constant), with a corresponding suitably varying choice of $c$ to ensure that $Pr[I_j = 1] \geq \frac{(\ln n)^{(b+2)}}{n}$.

Also, if $E[X] \to \gamma > 0$, the asymptotic probability of some point being uncovered is a positive finite quantity, and the condition is still necessary for asymptotic coverage probability to approach 1.

*b) $p \geq 1 - \frac{1}{n^{1+\varepsilon}} (0 < \varepsilon < 1)$:* We obtain that $Pr[\text{ no nodes alive }] = p^n \geq \left(1 - \frac{1}{n^{1+\varepsilon}}\right)^n$. As $n \to \infty$, the following holds:

$$\lim_{n\to\infty} Pr[\text{some point not covered}] \geq Pr[\text{no node alive}] \tag{57}$$

$$= \lim_{n\to\infty} \left(1 - \frac{1}{n^{1+\varepsilon}}\right)^n = e^{-\lim(\frac{1}{n^{\varepsilon}})} = 1 \text{ from Fact 2} \tag{58}$$

Thus the network is trivially not covered, regardless of transmission range.

## XVI. Sufficient Condition for Connectivity and Coverage

We now present a sufficient condition for the asymptotic existence of both connectivity and coverage.

**THEOREM 10.** *When $d \geq 32 \frac{\ln n}{\ln \frac{1}{p}}$, the network is asymptotically connected and covered with probability 1.*

*a) $p \leq \frac{1}{n^{1+\varepsilon}}$:* When the failure probability is so small as to fall in this range, the probability of even a single node failing approaches 0 asymptotically, and thus connectivity and coverage is trivially ensured even with the minimum transmission range of 1. This may be seen thus:

$$Pr[\text{No failures;full connectivity/coverage}] = (1-p)^n \tag{59}$$

$$\geq \left(1 - \frac{1}{n^{1+\varepsilon}}\right)^n \tag{60}$$

$$\lim_{n\to\infty} Pr[\text{No failures;full connectivity/coverage}] \tag{61}$$

$$\geq \lim_{n\to\infty} \left(1 - \frac{1}{n^{1+\varepsilon}}\right)^n = e^{-\lim(\frac{1}{n^{\varepsilon}})} = 1 \text{ from Fact 2} \tag{62}$$

*b) $p = \Omega(\frac{1}{n})$:* *Proof:*

Consider the subdivision of the grid as depicted in Fig. 7, so that the resulting cells have x-extents (y-extents) 0 to $a$, $a+1$ to $a+b$, $a+b+1$ to $2a+b+1$, and so on. Here $a = \lfloor \frac{r}{2} \rfloor$ and $b = r - a = r - \lfloor \frac{r}{2} \rfloor$. Then, each node is within range of all other nodes in the cells adjoining its own. Thus it is obvious that if each square has at least one non-faulty node, there exists a connected backbone that covers all points, and hence all nodes. Thus all non-faulty nodes are connected to each other via this backbone. The dimensions of the cells thus obtained can be $(a+1)^2$, $(a+1)b$ or $b^2$. Thus the population $k$ of any cell satisfies $k \geq \frac{r^2}{4}$, and the maximum possible number of cells $m \leq \frac{4n}{r^2}$. Then:

$$Pr[\text{ at least one node alive in a given cell }] = 1 - p^k \geq 1 - p^{\frac{r^2}{4}} \tag{63}$$

$$\therefore Pr[\text{ at least 1 node alive in each cell}] \geq \left(1 - p^{\frac{r^2}{4}}\right)^{\frac{4n}{r^2}} \tag{64}$$

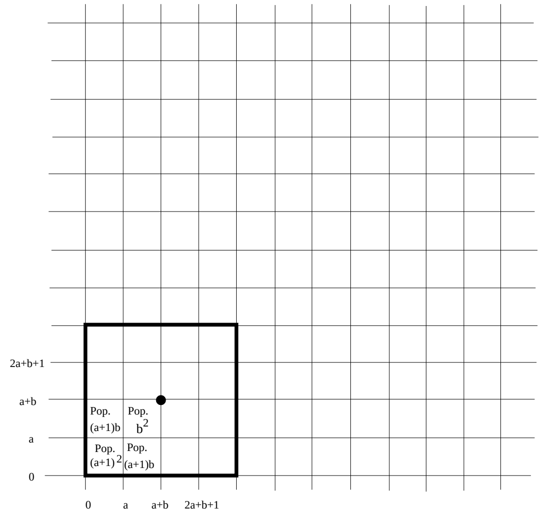


Fig. 7. Subdivision of network into cells

Let us choose $r \geq \sqrt{\frac{8\ln n}{\ln\frac{1}{p}}}$. Then:

$$Pr[\text{at least 1 node alive in each cell}] \geq \left(1 - p^{\frac{r^2}{4}}\right)^{\frac{n\ln\frac{1}{p}}{2\ln n}} \tag{65}$$

Since $p \geq \alpha\frac{1}{n}$ for some constant $\alpha$ , $\ln\frac{1}{p} \leq \ln n - \ln\alpha$. Hence:

$$Pr[\text{at least 1 node alive in each cell}] \geq \left(1 - p^{\frac{r^2}{4}}\right)^{\frac{n\ln\frac{1}{p}}{2\ln n}} = \left(1 - p^{\frac{2\ln n}{\ln\frac{1}{p}}}\right)^{\frac{n\ln\frac{1}{p}}{2\ln n}} \geq \left(1 - \frac{1}{n^2}\right)^{\frac{n}{2}(1-\frac{\ln\alpha}{\ln n})} \tag{66}$$

$$\tag{67}$$

Thus, by application of Fact 2, we obtain:

$$\lim_{n\to\infty} Pr[\text{at least 1 node alive in each cell}] \geq \lim_{n\to\infty}\left(1 - \frac{1}{n^2}\right)^{\frac{n}{2}(1-\frac{\ln\alpha}{\ln n})} = e^{-\lim(\frac{1}{2n})} = 1 \text{ from Fact 2} \tag{68}$$

Since this condition ensures connectivity and coverage, we obtain that:

$$\lim_{n\to\infty} Pr[\text{network is connected and covered}] \to 1 \tag{69}$$

## XVII. CONDITIONS IN EUCLIDEAN METRIC

We show that our results derived for $L_\infty$ metric continue to hold for $L_2$ metric, with only the constants in the theta notation changing.

**LEMMA 8.** *If the network is asymptotically connected (covered) in $L_\infty$ for all $r \geq r_{min}$, then the network is connected (covered) asymptotically in $L_2$ for all $r \geq r_{min}\sqrt{2}$.*

*Proof:* The proof is by contradiction. Suppose that, for a given failure configuration, the network is asymptotically connected in $L_\infty$ for all $r \geq r_{min}$ but is not asymptotically connected for all $r \geq r_{min}\sqrt{2}$

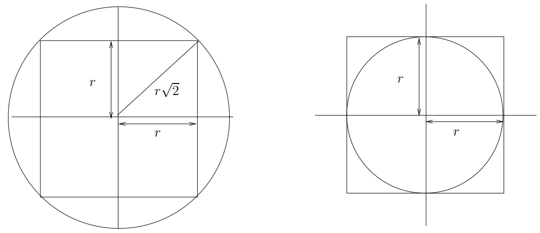


Fig. 8. Relationship between $L_\infty$ and $L_2$ neighborhoods

in $L_2$. Observe that it is possible to circumscribe a $L_\infty$ neighborhood of range $r$ by a $L_2$ neighborhood of range $r\sqrt{2}$ (Fig. 8). Hence the nodes in an $L_2$ network of transmission range $r\sqrt{2}$ can be made to simulate the operation of nodes in a $L_\infty$ network with range $r$ (as the $L_\infty$ neigborhood is fully contained within the $L_2$ neighborhood). This implies that if the $L_\infty$ network of range $r$ is connected (covered), so must be the $L_2$ network of range $r\sqrt{2}$. If there is some $r \geq r_{min}$ for which the $L_\infty$ network of range $r$ is connected (covered) asymptotically, but the $L_2$ network of range $r\sqrt{2}$ is not, we obtain a contradiction, as connectedness (coverage) of the $L_\infty$ network would imply connectedness (coverage) of the $L_2$ network.

**LEMMA 9.** *If the network is asymptotically disconnected (not covered) in $L_\infty$ for all $r \leq r_{min}$, then the network is disconnected (not covered) asymptotically in $L_2$ for all $r \leq r_{min}$.*

*Proof:* The proof is by contradiction. Suppose that the network is asymptotically disconnected (not covered) in $L_\infty$ for range $r$, but is not disconnected (not covered) in $L_2$ for range $r$. Observe that an $L_\infty$ neighborhood of transmission range $r$ circumscribes an $L_2$ neighborhood of range $r$ (Fig. 8). Thus, for any given random failure configuration, if the $L_2$ network of range $r$ were connected (covered), so would be the $L_\infty$ network of radius $r$, as we could simply make the nodes in the $L_\infty$ network simulate the behavior of nodes in the $L_2$ network, and obtain connectedness (coverage). Hence, if the $L_2$ network of range $r \leq r_{min}$ is not asymptotically disconnected (not covered), the $L_\infty$ network of range $r \leq r_{min}$ must also not be disconnected (not covered). This yields a contradiction.

## XVIII. Discussion

It is interesting to note that in case of a grid network, the necessary and sufficient node degree turns out to be $\Theta\left(\frac{\log n}{\log \frac{1}{p}}\right)$, as compared to $\Theta(\frac{\log(n(1-p))}{1-p})$ (when expressed in our notation) for the case of a randomly deployed network, where sensors are active with probability $1-p$ [14]. However, it is not difficult to see that such a difference is to be expected. In a grid network, as failure(or sleep) probability $p \rightarrow 0$, the network tends towards a deterministic topology, whereas in a random network, if failure or sleep probability $p \rightarrow 0$, the network can only tend towards a denser but still random network. Thus, at small values of $p$, a very small degree will suffice for a grid network, but may not for a random network. At larger $p$ values, the grid network exhibits increasing randomness and begins to resemble a network with random deployment. Thus, one may see that the two expressions are within a small range of each other when $p$ is large (given sufficiently large $n$), but diverge as $p \rightarrow 0$.

Another observation is that the form of the results is very similar to results obtained by us for reliable broadcast in a grid network with Byzantine failures. For Byzantine failures, we have obtained that the necessary and sufficient conditions for reliable broadcast entail a node degree of $\Theta\left(\frac{\ln n}{\ln \frac{1}{2p} + \ln \frac{1}{2(1-p)}}\right)$, which may be re-stated as $\Theta\left(\frac{\ln n}{D(Q_{\frac{1}{2}}||P)}\right)$ where $Q_{\frac{1}{2}}$ denotes a distribution with failure probability $\frac{1}{2}$, $P$ denotes the actual distribution with failure probability $p$, and $D(Q||P)$ denotes the *relative entropy* (or Kullback-Leibler distance) between distributions $Q$ and $P$. Similarly, one may view the node degree for connectivity as $\Theta\left(\frac{\ln n}{\lim_{q\to 1} D(Q||P)}\right)$, where $Q$ is the distribution with failure probability $q$, and $P$ is the actual failure distribution.

## XIX. Non-Toroidal Networks

We have made the assumption that the network is toroidal, so as to avoid edge effects. However, we can see that the degree of any node at the outermost edge is no more than $d$, and at least $\frac{d}{4}$ (where $d$ is the uniform degree that each node would have in the toroidal case). Thus, the necessary condition would continue to hold as is (since some nodes having a lesser degree can only increase the probability of disconnection). The construction used to prove the sufficient condition also continues to hold as is, since all full-cells in the tiling will have at least one active node each, and even if there are regions at the fringes left-over, they will still fall within range of some active node in the nearest full tile (due to the chosen dimensions of the cells). Thus, the results are not affected. A similar argument leads to the conclusion that the coverage results are not affected.

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