On Reliable Broadcast in a Radio Network: A Simplified Characterization

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Vartika Bhandari
Dept. of Computer Science, and
Coordinated Science Laboratory
University of Illinois at Urbana-Champaign
vbhandar@uiuc.edu

Nitin H. Vaidya
Dept. of Electrical and Computer Eng., and
Coordinated Science Laboratory
University of Illinois at Urbana-Champaign
nhv@uiuc.edu

***Abstract*— We consider the problem of reliable broadcast in an infinite (or finite toroidal) radio network under Byzantine and crash-stop failures. We present a simpler characterization and proofs for results proved earlier in [1].**

I. Introduction

This paper augments [1] by presenting a simpler characterization of a sufficient condition for achieving reliable broadcast. We first state and justify the condition for a general graph. Thereafter we focus on the grid network model for which results were presented in [1]. We present a simpler construction for these results, based on the new characterization.

II. The Radio Network Model

We assume a perfectly reliable wireless channel (and absence of address-spoofing or collisions) such that, if a node transmits a value, all its neighbors hear the transmission, and are certain of the identity of the sender. The transmitting node is thus incapable of duplicity, because if it were to attempt sending contradicting messages, they would be heard by all its neighbors, and its duplicity would be detected. Thus any protocol could stipulate that if the neighbors of a node hear it transmitting multiple contradictory versions of a message, they should accept only the first message, and ignore the rest.

III. Notation and Terminology

Nodes are identified by their grid location i.e. (x,y) denotes the node at (x,y). The neighborhood of (x,y) comprises all nodes within distance r of (x,y) and is denoted as $nbd(x,y)$. For succint description, we define a term $pnbd(x,y)$ where $pnbd(x,y) = nbd(x-1,y) \cup nbd(x+1,y) \cup nbd(x,y-1) \cup nbd(x,y+1)$. Intuitively $pnbd(x,y)$ denotes the *perturbed neighborhood* of (x,y) obtained by perturbing the center of the neighborhood to one of the nodes immediately adjacent to (x,y) on the grid. We shall occasionally refer to $nbd(S)$ where S is a set. In such cases, $nbd(S) = \bigcup_{x \in S} nbd(x)$. Besides, throughout this paper, a non-faulty node may be referred to as an honest or correct node. A faulty node that exhibits byzantine failure shall occasionally be referred to as a malicious node.

IV. A General Sufficient Condition

Consider a general graph $G = (V,E)$. Designate a source $s \in V$ as the source of the broadcast. Then a s-cut is a partition $C = (S, V-S)$ such that $s \in S$. S can potentially denote the set of nodes that have already had the opportunity to correctly determine the broadcast value, and commit to it (note that all correct nodes in S will thus indeed have committed to the correct value, while the behavior of faulty nodes is indeterminate). $V-S$ can potentially denote the set of nodes that are yet to do so.

Let us first consider the case where G is a finite graph. Then any cut C may be considered as an envelope for the advancing frontier of the broadcast at some instant. If the cut C were indeed encountered during algorithm operation, this holds trivially. However, even if the cut $C = (S, V-S)$ were not actually encountered during algorithm operation, the following holds:

At any point of time t during algorithm operation, let the actual frontier be denoted by the cut $C_{actual}(t) = (S_{actual}(t), V - S_{actual}(t))$. Consider an algorithm step at time t' such that at $t < t'$, $S_{actual}(t) \subseteq S$, and $S_{actual}(t') \not\subseteq S$. Thus at time t', at least one node $u \in V-S$ crossed over from $V - S_{actual}$ to S_{actual}. At time $t < t'$, the frontier of the broadcast (i.e. C_{actual}) lay strictly behind the frontier defined by $C = (S, V-S)$. Thus, if a node has sufficient information flowing to it from S_{actual} to be able to cross-over, then it must necessarily have at least as much information flowing to it from S, and be able to cross the cut $C = (S, V-S)$ were it ever encountered. This is depicted in Fig. 1. Thus, the following two statements are equivalent:

- *Statement 1:* For every s-cut $(S, V-S)$ of the graph that is actually encountered during algorithm execution, some node $u \in V-S$ possesses sufficient connectivity to be able to cross-over to S from $V-S$.
- *Statement 2:* For every possible s-cut $(S, V-S)$ of the graph, assuming all nodes in S have had the opportunity to

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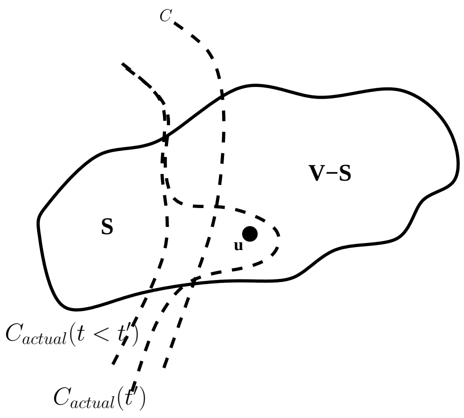

Fig. 1. Equivalence of Cut Conditions

make a correct determination (and correct nodes actually have made it), some node $u \in V - S$ possesses sufficient connectivity to be able to cross-over to S.

Thus we see that Statement 2 does not impose a more stringent requirement than Statement 1 for a finite graph.

Lemma 1: Given a finite graph G, Statement 1 is a sufficient condition for feasibility of broadcast, and Statement 2 is thus an equivalent sufficient condition.

Proof: This may be seen as follows: since Statement 1 holds for every encountered cut, the set $V - S$ will monotonically decrease over time, and being finite will eventually become empty. At that stage $S = V$, and the broadcast will have successfully reached every node. Statement 2 is equivalent to Statement 1, and is hence also a sufficient condition. ■

If G is instead an infinite graph, the equivalence does not hold. Besides, Statement 1 ceases to be a sufficient condition. However, the following still holds:

Lemma 2: Given an infinite graph $G = (V,E)$, Statement 2 is a sufficient condition for feasibility of broadcast.

Proof: This may be established via contradiction: suppose Statement 2 holds for graph $G = (V,E)$, but some nodes are incapable of determining the correct broadcast value. Consider the set D comprising all such nodes (note that D may be an infinite set). Consider the corresponding cut $((S = (V - D)), D)$. If Statement 2 holds, then $\exists u \in D$ that should be able to make a correct determination, and cross over to $S = (V - D)$. Since all nodes in $V - D$ are capable of determining the correct value, and those in D are not, the broadcast status of the graph must asymptotically tend to $(V - D, D)$. However, when this cut is encountered, there would be some node capable of crossing over from D to $V - D$. That would violate the assumption that $(V - D, D)$ represents the asymptotic broadcast status. Thus Statement 2 is indeed a sufficient condition.

■

Note that the cut $(V - D, D)$ may never actually be encountered during algorithm execution, as $V - D$ may be infinite, and thus the frontier could continue to expand forever without ever encountering this cut. This explains why Statements 1 and 2 are not equivalent for infinite graphs.

It now remains to characterize what constitutes sufficient connectivity to be able to cross-over to the source side of the cut. The goal of any reliable broadcast algorithm is that each node should be able to eventually decide on the correct broadcast value. If at any instant, the frontier is represented by cut $C = (S, V - S)$, then by the assumption of Statement 2, all nodes in S have correctly determined the broadcast value. Any communication of information across the cut must happen through the nodes comprising the cut-vertices viz. $C_S = \{v \in S | \exists (v,u) \in E \wedge u \in V - S\}$. Thus for the purpose of analysis, it suffices to reduce the source side of the cut S to $S' = s_{sup} \cup C_S \cup (nbd(C_S) \cap S)$, with s_{sup} being a new *super-source* node that acts as an abstract embodiment of the original source, and is connected directly to each node in C_S (via the pseudo-edges). The neighbors of the cut-vertices nodes on the source side are included to enforce the per-neighborhood fault constraint amongst the cut-vertices nodes. We refer to the corresponding graph induced by $S' \cup (V - S)$, with the pseudo-edges added, as the reduced graph G'.

For a given graph $G = (V,E)$, it suffices to show that for each s-cut $C = (S, V - S)$ of G, some node $u \in V - S$ possesses sufficient connectivity to s_{sup} be able to determine the correct value and thus cross over to S. We state and prove the following sufficient condition:

THEOREM 1: Given a graph $G = (V,E)$ and designated source s, with upto t byzantine faults in any neighborhood, reliable broadcast is possible in G if every s-cut $C = (S, V - S)$ (with cut-vertices C_S) satisfies the following:
$\exists u \in V - S$ such that either $(s,u) \in E$ or $\exists (2t + 1)$ node-disjoint (s_{sup}, u) paths in the reduced graph G', such that all intermediate nodes on these paths lie within the neighborhood of some single node $v \neq s_{sup}$.

Proof: Since all nodes in S, and hence $C_S \subseteq S$, have had the opportunity to correctly determine the broadcast value (by assumption), the addition of pseudo-edges with s_{sup} ensures this same property (since neighbors of the source can trivially determine the value correctly), while removing from consideration nodes that are no longer relevant to the result we seek to prove. If a node is connected to s_{sup} via at least $2t + 1$ node-disjoint paths that all lie within some single neighborhood, then at most t of these paths may be faulty (as no more than t faults may exist in any single neighborhood). Thus, the node u will receive the correct value over at least $t + 1$ paths, and will be in a position to commit to it. The situation is illustrated in Fig. 2.

By Lemma 1, this is a sufficient condition for finite

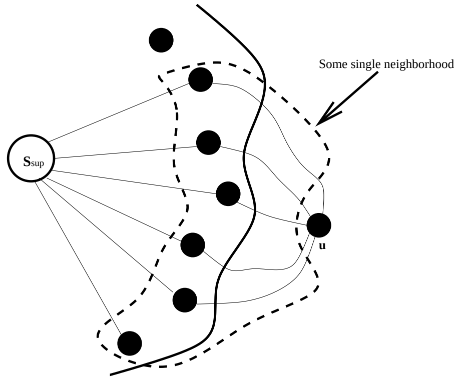

Fig. 2. Connectivity to super-source

graphs. By Lemma 2, this is a sufficient condition for infinite graphs. ■

Corollary 1: Given a graph $G = (V,E)$ and designated source s, with upto t crash-stop faults in any neighborhood, reliable broadcast is possible in G if every s-cut $C = (S, V - S)$ (with cut-vertices C_S) satisfies the following:
$\exists u \in V - S$ such that either $(s,u) \in E$ or $\exists (t + 1)$ node-disjoint (s_{sup}, u) paths in the reduced graph G', such that all intermediate nodes on these paths lie within the neighborhood of some node $v \neq s_{sup}$.

Proof: When crash-stop failures are considered, reachability is synonymous with achievability of reliable broadcast. If a node is connected to $t+1$ nodes in S via one path each such that all $t+1$ paths are node-disjoint, and lie in a single neighborhood, then at most t of these can be faulty. Thus there will be at least one fault-free path through which the node may be reached, and broadcast can propagate. ■

Corollary 2: Given a graph $G = (V,E)$ and designated source s, if the sufficient condition in Theorem 1 is satisfied by G for t byzantine faults in any neighborhood, then $2t$ crash-stop faults in any neighborhood can be tolerated.

Proof: Proceeds from Theorem 1 and Corollary 1. ■

V. Grid Network Model

Nodes are located on an infinite grid (each grid unit is a 1×1 square). Nodes can be uniquely identified by their grid location (x,y). All nodes have a transmission radius r. A message broadcast by a node (x,y) is heard by all nodes within distance r from it (where distance is defined in terms of the particular metric under consideration, and r is assumed to be an integer). The set of these nodes is termed the neighborhood of (x,y). Thus there is an assumption that the channel is perfectly reliable, and a local broadcast is correctly received by all neighbors. We call this the *reliable local broadcast* assumption. In this paper, we consider two distance metrics viz. L_∞ and L_2. The L_∞ metric is essentially the metric induced by the L_∞ norm [2], such that the distance between points (x_1,y_1) and (x_2,y_2) is given by $\max\{|x_1 - x_2|, |y_1 - y_2|\}$ in the this metric. Thus $nbd(a,b)$ comprises a square of side $2r$ with its centroid at (a,b). The L_2 metric is induced by the L_2 norm [2], and is the Euclidean distance metric. The L_2 distance between points (x_1,y_1) and (x_2,y_2) is given by $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$, and $nbd(a,b)$ comprises nodes within a circle of radius r centered at (a,b).

As in [3], we assume that a node may not spoof another node's identity, and that no collisions are possible, i.e., there exists a pre-determined TDMA schedule that all nodes follow. Such schedules are easily determined for the grid network under consideration, e.g., the schedule in [3] (so long as time-optimality is not a concern). A designated source (assumed located at $(0,0)$ w.l.o.g.) broadcasts a message with a binary value.

VI. Reliable Broadcast with Byzantine Failures

We prove the following:

THEOREM 2: If $t < \frac{1}{2}r(2r+1)$, reliable broadcast is achievable in the L_∞ metric.

This was earlier established in [1] using a stronger connectivity condition. We now prove the same using a weaker connectivity requirement that is essentially the same as the general sufficient condition of Theorem 1. A brief comparative discussion is presented in Section VI-A. We present a protocol to achieve reliable broadcast, based on the weaker condition. Without loss of generality we assume the message to comprise a binary value (say 0 or 1). A non-faulty node that is not the source is said to *commit* to a value when it becomes certain that it is indeed the value originated by the source. The protocol requires maintenance of state by each node pertaining to messages received from nodes within its two-hop neighborhood. This state may be reduced further by stipulating exact messages that a node should lookout for, and shall be evident from our constructive proof for the viability of reliable broadcast with $t < \frac{1}{2}r(2r+1)$. However, at a basic level, the protocol operates as follows:

- Initially, the source does a local broadcast of the message.
- Each neighbor i of the source commits to the first value it heard from the source and does a one-time local broadcast of a $COMMITTED(i,v)$ message.
- Hereafter, the following protocol is followed by each node j (including those involved in the previous two steps):

  On receipt of a $COMMITTED(i,v)$ message from neighbor i, record the message, and broadcast a $HEARD(j,i,v)$ message.

  On receipt of a $HEARD(j,i,v)$ message, record the message, but do not re-propagate.

  When a node j commits to a value v, it does a one-time local broadcast of a $COMMITTED(j,v)$ message.

A node P commits to a value v when it is *certain* about it. A node is said to be *certain* about a value v if it receives v through *COMMITTED* or *HEARD* messages over at least $t+1$ node-disjoint paths that lie within a single neighborhood. More precisely, a node P is *certain* of a value v if $\exists Q \in V, C \subseteq V$ such that $C \subseteq nbd(Q)$, and P received some $t+1$ messages $m_1, m_2, ..., m_{t+1}$ such that $m_i = COMMITTED(a_i, v)$ or $m_i = HEARD(a_i, a_{i'}, v)$, $\forall i, a_i, a_{i'} \in C$ and $\forall i, j, a_i \neq a_j, a_i \neq a_{j'}$.

THEOREM 3: (Correctness) No node shall commit to a wrong value by following the above rule.

Proof: The proof is by contradiction. Consider the first node, say j, that makes a wrong decision to commit to value v. This implies it received the value v from at least $t+1$ nodes through a single path (direct or two-hop) each, such that all $t+1$ paths were node-disjoint, and lay in some single neighborhood. Since the number of faults in any single neighborhood may be at most t, it implies that at most t of these paths could have a faulty source (of a *COMMITTED* message) or a faulty intermediate node (that sends a *HEARD* message). Thus, all paths cannot have relayed the wrong value, and so v must indeed be the correct value. ■

THEOREM 4: (Completeness) Each node is eventually able to commit to the correct value.

Proof: We prove that each node will be able to meet the conditions stipulated by the protocol for committing to the correct value. The proof also clarifies the operation of the protocol, and in fact would allow one to stipulate exactly which messages each node should act upon (given that the topology of the network is completely known), thereby reducing the state maintained at each node. The essence of the proof lies in showing that each node P (except the direct neighbors of $(0,0)$) has $(2t+1)$ node-disjoint paths lying in some single neighborhood into a part of the network that has already committed to the correct value. This is akin to the general sufficient condition of Theorem 1.

The proof proceeds by induction.

Base Case:

All honest nodes in $nbd(0,0)$ are able to commit to the correct value. This follows trivially since they hear the origin directly, and we assume that address-spoofing is impossible.

Inductive Hypothesis:

If all honest neighbors of a node located at (a,b) i.e. all honest nodes in $nbd(a,b)$ are able to commit to the correct value, then all honest nodes in $pnbd(a,b)$ are able to commit to the correct value.

Proof of Inductive Hypothesis:
We show that for each node P in $pnbd(a,b) - nbd(a,b)$ there exists a set of 2t+1 paths $\{P_1, P_2, ..., P_{2t+1}\}$ of the form $P_i = (A_i, P)$ or $P_i = (A_i, A'_i, P)$, such that all A_i, A'_i are distinct, lie in some single neighborhood, and $A_i \in nbd(a,b)$. Since no more than t of the A_i, A'_i can be faulty, this guarantees that the node will receive the correct value through at least $(t+1)$ paths, and will also commit to it.

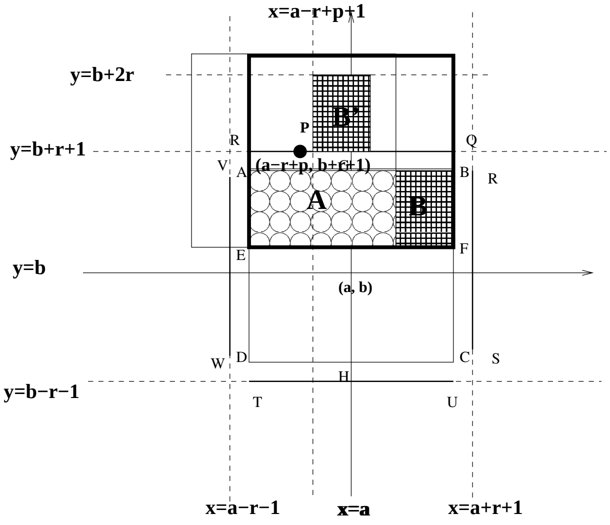

Fig. 3. Existence of Sufficient Connectivity

Consider a node P belonging to $nbd(a,b+1)$. The analysis for nodes in $nbd(a,b-1), nbd(a-1,b), nbd(a+1,b)$ is similar.

Node P in $nbd(a,b+1) - nbd(a,b)$ may be considered to be located at $(a-r+p, b+r+1)$ where $\{0 \leq p \leq 2r\}$ (Fig. 3). We show analysis of locations of P corresponding to $\{0 \leq p \leq r\}$. A similar argument holds for the remaining locations, by virtue of symmetry. We show the existence of $r(2r+1)$ node-disjoint paths $P_1, P_2, ..., P_{r(2r+1)}$, that all lie within the same single neighborhood (centered at $(a, b+r+1)$, and indicated by the square with dark outline in Fig. 3). The region marked A comprises $\{(x,y) | (a-r) \leq x \leq (a+p); (b+1) \leq y \leq (b+r)\}$, and nodes in this region lie in $nbd(a,b)$, and are neighbors of P. Thus, there are $r(r+p+1)$ paths of the form $A \rightarrow P$. The region B comprises $\{(x,y) | (a+p+1) \leq x \leq (a+r); (b+1) \leq y \leq (b+r)\}$, and falls in $nbd(a,b)$. The region B' comprises $\{(x,y) | (a+p+1-r) \leq x \leq a; (b+r+1) \leq y \leq (b+2r+1)\}$, and falls in $nbd(P)$. As may be seen, B' is obtained by a translation of B to the left by r units, and then up by r units. Thus there is a one-to-one correpondence between a point (x,y) in B and a point $(x-r, y+r)$ in B', such that the points in each pair are neighbors. This yields $r(r-p)$ paths of the form $B \rightarrow B' \rightarrow P$.

Thus the $r(2r+1)$ node-disjoint paths are obtained.

Observe that the inductive hypothesis along with the base case suffice to show that every non-faulty node will eventually commit to the correct message, since starting at $(0,0)$, one can cover the entire infinite grid by moving up, down, left and right. Thus the neighborhood of every grid point can be shown to have decided i.e. every non-faulty node will have decided on the correct value.

We note that the connectivity condition proved above is also sufficient to prove that upto $2t < r(2r+1)$ crash-stop failures are tolerable in L_∞ metric. We shall elaborate further in Section VII. ■

A. Comparison with Earlier Proof

The earlier proof for the possibility bound [1] was based on the much stronger condition that every node in $pnd(a,b) - nbd(a,b)$ has $2t+1$ node-disjoint paths, all lying within some single neighborhood, to each of $2t+1$ nodes in $nbd(a,b)$. However, as discussed earlier, a much weaker condition suffices to ensure reliable broadcast. The resultant protocol is also more efficient in terms of greater localization of propagated messages. The earlier proof is still of interest, as a general statement about connectivity properties of the grid network under consideration. The proved connectivity property may also find use in distributed operations other than reliable broadcast.

VII. Crash-Stop Failures

When only crash-stop failures are admissible, no special protocol is required. Each node that receives a value, commits to it, re-broadcasts it once for the benefit of others, and then may terminate local execution of the protocol. Thus the sole criterion for achievability is reachability. In this failure mode, we established an exact threshold for tolerable faults in L_∞ metric [1]. The same may be proved using the connectivity condition for byzantine failures described in the previous section.

THEOREM 5: If $t < r(2r+1)$, it is possible to achieve reliable broadcast in L_∞ metric.

Proof: Consider the proof for the byzantine protocol. Given that $nbd(a,b)$ has decided, there exist $r(2r+1)$ node-disjoint paths of the form described in Theorem 4 that lie in one single neighborhood. Since $t < r(2r+1)$, at least one path will be fault-free, thereby enabling the broadcast to propagate to $pnbd(a,b)$. Thus, by inductive reasoning, all fault-free nodes on the grid will receive the broadcast. ■

VIII. Reliable Broadcast in Euclidean Metric

We now briefly consider the issue of reliable broadcast in the L_2 i.e. Euclidean metric. As in [1], we refrain from establishing exact thresholds as it is difficult to precisely determine lattice points falling in areas bounded by circular arcs. We present informal arguments based on the new (and simpler) characterization to suggest that reliable broadcast in L_2 is achievable if slightly less that one-fourth fraction of nodes in any neighborhood exhibit Byzantine faults. We work with the value $t < 0.24\pi r^2$. The basis for the argument is that for sufficiently large r, the number of nodes that lie in various considered subregions (having area A) of a circle of radius r (elaborated later) are approximately $A \pm O(r)$. Thus, we expect the argument to hold well for large values of r. The argument proceeds by induction, as in the previous section.

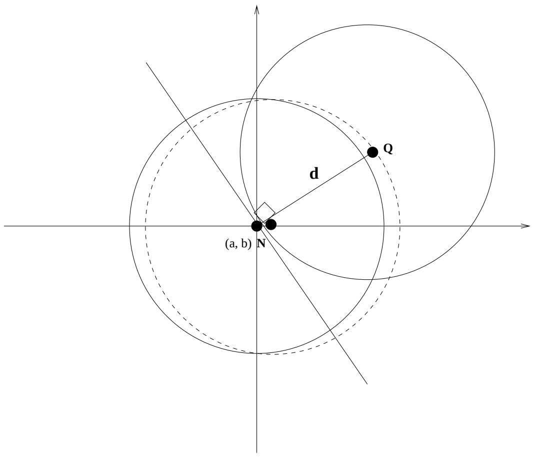

Fig. 4. Illustrating an Approximate Argument for Euclidean Metric

Base Case:

All honest nodes in $nbd(0,0)$ are able to commit to the correct value. This follows trivially since they hear the origin directly.

Inductive Hypothesis:

If all honest neighbors of a node located at (a,b) are able to commit to the correct value, then all honest nodes in $pnbd(a,b)$ are able to commit to the correct value.

Justification of Inductive Hypothesis:
We show that each node in $pnbd(a,b) - nbd(a,b)$ is connected to $2t+1$ nodes in $nbd(a,b)$ via one path each, such that all these $2t+1$ paths are node-disjoint and they all (the endpoints, as well as any intermediate nodes) lie entirely in one single neighborhood. Since no more than t of these can be faulty, this would guarantee that the node will receive the correct value through at least $t+1$ such paths, and commit to it.

Consider the node at (a,b), as in Fig. 4. Let d be the distance between the node at (a,b) (we call it node N) and any node in $(pnbd(a,b) - nbd(a,b))$ (we call it node Q). Then $d \leq r+1$ (by the triangle inequality). We consider the situation in Fig. 5 with NQ from Fig. 4 rotated to the horizontal axis. We attempt to construct node-disjoint paths that all lie within the neighborhood centred at M (the midpoint of NQ) or the grid location nearest to it. If M is itself not a grid point, the resultant perturbation of the neighborhood centre to the nearest grid location can only

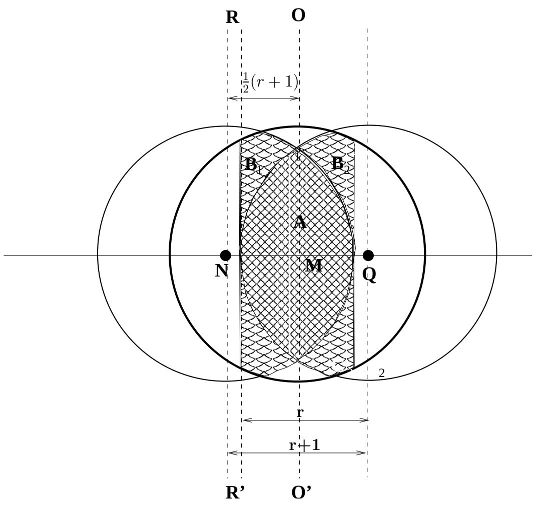

Fig. 5. Approximate Construction depicting Node-Disjoint Paths (NQ from Fig. 4 rotated to x-axis)

affect the presented calculations by $O(r)$. The set of nodes marked A are common neighbors of P and Q and constitute one-hop paths ($A \rightarrow Q$). A set of two-hop paths $B_1 \rightarrow B_2 \rightarrow Q$ is also formed where each point (x,y) in region B_1 has a corresponding point in B_2 (its image under reflection by axis OO'). The number of paths is approximately equal to the sum of the areas A and B_1 which turns out to be approximately $1.538r^2 = 0.49\pi r^2 > (2(0.24\pi r^2)+1)$ (for sufficiently large r). The details of the calculation are presented in the appendix. Thus approximately $0.24\pi r^2$ Byzantine faults may be tolerated.

Observe that the above argument also leads to the conclusion that upto $2t = 0.48\pi r^2$ crash-stop failures may be tolerated.

Appendix

Calculation of Collective Area of regions A and B_1 from Section VIII.

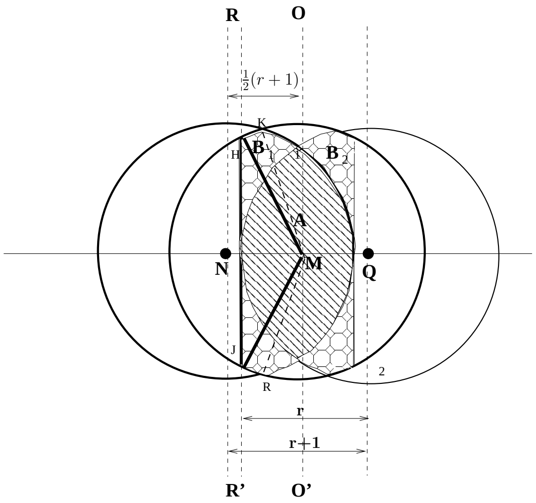

Fig. 6. Calculation of Collective Area of Regions A and B_1 (from Fig. 5)

Consider Fig. 6. Then the collective area of regions A and B_1 = Area of $nbd(N) \cap nbd(M)$ - Area of Sector HMJ + Area of $\triangle HMJ$. We show the calculations below. All angles are in radians. Sector KMR (HMJ) or $\triangle$ KMR (HMJ) refers to the sector/triangle subtending obtuse (and not reflex) angle KMR (HMJ) at M.

Area of $nbd(N) \cap nbd(M)$ = 2 (Sector KMR - $\triangle$ KMR).

Area of Sector KMR = $\pi r^2 \frac{\angle KMR}{2\pi} = \pi r^2 \frac{(2cos^{-1}(\frac{r+1}{4r})))}{2\pi} \approx (r^2(cos^{-1}(\frac{1}{4}))) \approx 1.318r^2$.

Area of $\triangle$ KMR = $\frac{1}{2}r^2\sin(\angle KMR) \approx 0.242r^2$. Thus Area of $nbd(N) \cap nbd(M) = 2(1.318-0.242)r^2 = 2(1.076)r^2 = 2.152r^2$.

Area of $\triangle HMJ = \frac{1}{2}r^2 sin(\angle HMJ) = \frac{1}{2}r^2\sin(2\cos^{-1}(\frac{r+1}{2r})) \approx 0.433r^2$.

Area of Sector HMJ = $\pi r^2 \cdot \frac{\angle HMJ}{2\pi} = 1.047r^2$.

Thus collective area of A and B_1 is give by:

$$2.152r^2 - 1.047r^2 + 0.433r^2 = 1.538r^2 \approx 0.49\pi r^2.$$

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