COSC 240 Introduction to Algorithms
Spring 2019
Problem Set 3
Due: February 19, 2019
Maximum points 40

1. (16 points) An undirected graph is said to be connected if there is a path between every pair of nodes in the graph. Consider the following algorithm for an undirected connected graph G(V,E). Assume that all edge weights are positive and distinct.

   (i) Find subgraphs $G_1(V_1,E_1)$ and $G(V_2,E_2)$ of G such that each subgraph is an undirected connected graph. No vertex belongs to both V_1 and V_2, and $V_1 \cup V_2 = V$. Set E_1 contains all edges in E that have both endpoints in V_1. Similarly, E_2 contains all edges in E that have both endpoints in V_2.
   (ii) Find spanning trees for subgraphs G_1 and G_2, respectively.
   (iii) To the edges in the above two spanning trees, add the least-weight edge in E that has one endpoint in V_1 and the other endpoint in V_2.

   (a) Does the set of edges obtained in step (iii) above form a spanning tree of graph G? If you answer yes, explain why. If you answer no, provide a counter-example.

   (b) If the spanning trees obtained in step (ii) above are *minimum* spanning trees for G_1 and G_2, respectively, does the set of edges obtained in step (iii) above form a *minimum* spanning tree of graph G? ? If you answer yes, explain why. If you answer no, provide a counter-example.

2. (8 points) Does Dijkstra's algorithm necessarily determine weights of all the shortest paths if no loops have negative weights, but some of the edge weights may be negative? If you answer yes, provide a justification. If you answer no, provide a counter-example of a graph containing at most 4 vertices.

3. (16 points) Consider Prim's spanning tree algorithm for an undirected graph. For the purpose of this question, we will say that edge (u,v) is in the spanning tree obtained by the algorithm provided that either $v=\pi[u]$ or $u=\pi[v]$ when the algorithm terminates.

   In each part below, state whether it is possible for the set of undirected edges in the above spanning tree to depend on which node is initially chosen to be node s in the algorithm. Provide a justification for your answer.

   (a) Assume that all edge weights are distinct.
   (b) Assume that the edge weights are not necessarily distinct.

   Hint: *Consider what may happen if the behavior of the priority queue may depend on the order in which keys may be decreased, particularly when two keys are decreased to the same value.*